Lecture 15: Weak convergence

δ- function: $\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1, \qquad \int_{-\infty}^{\infty} \delta(x) f(x)\,dx = f(0)$$

Fourier representation:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(x) e^{-ikx}\,dx = \frac{1}{2\pi}$$

This means $\delta(x) \sim \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{ikx}$

$$\sim \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \cos kx$$

Doesn't "converge" in traditional sense!

In what sense does this converge to δ-function? Weakly

Partial sums:

$$s_n(x) = \frac{1}{2\pi} + \sum_{k=1}^{n} \frac{1}{\pi}\cos kx$$

We can sum this!

$$s_n(x) = -\frac{1}{2\pi} + \mathrm{Re} \sum_{k=0}^{n} \frac{1}{\pi} \left(e^{ix}\right)^k$$

$$= -\frac{1}{2\pi} + \frac{1}{\pi} \mathrm{Re} \left(\frac{1 - e^{i(n+1)x}}{1 - e^{ix}} \right)$$

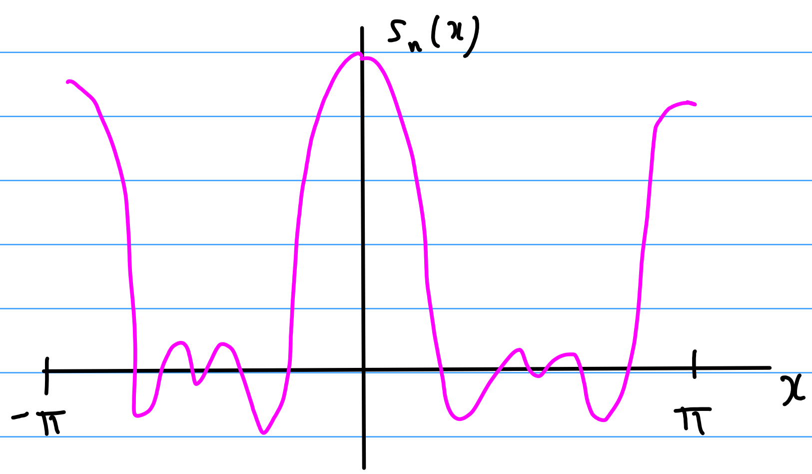
$$= -\frac{1}{2\pi} + \frac{1}{\pi} \mathrm{Re} \left(\frac{1 - e^{i(n+1)x}}{-e^{ix/2}\, 2i \sin(x/2)} \right)$$

$$= -\frac{1}{2\pi} - \frac{1}{\pi \sin(x/2)} \mathrm{Re} \left(\frac{e^{-ix/2} - e^{i(n+\frac{1}{2})x}}{2i} \right)$$

$$= \cancel{-\frac{1}{2\pi}} - \frac{1}{\pi \sin(x/2)} \left(\cancel{-\frac{1}{2}\sin\left(\frac{x}{2}\right)} - \frac{1}{2} \sin\left(n+\frac{1}{2}\right)x \right)$$

$$\boxed{s_n(x) = \frac{1}{2\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}}$$

Dirichlet kernel



As $n \to \infty$, oscillate faster and faster.

Note that $\int_{-\pi}^{\pi} s_n(x)\,dx = 1$

Small mistake in book (6.40)

Also $\lim_{n\to\infty} s_n(0) = \infty$.

At each point $x \neq 0$, does converge to 0, so not a δ-function

i.e. $s_n(\pi/2) = \dfrac{1}{2\pi} + \dfrac{1}{\pi}\sum_{k=1}^{n}(-1)^n$

oscillates

However, converges weakly:

A sequence $f_n(x)$ converges weakly to $f_*(x)$ on $[a,b]$ if for every continuous test function $u(x) \in C^0[a,b]$ we have

$$\int_a^b f_n(x)\,u(x)\,dx \longrightarrow \int_a^b f_*(x)\,u(x)\,dx$$

as $n\to\infty$.

We write $f_n(x) \rightharpoonup f_*(x)$.

Let us prove that $\cos nx \rightharpoonup 0$ on $[-\pi,\pi]$

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} \cos nx\; u(x)\,dx = 0$$

Riemann-Lebesgue Lemma

This is easy if $u(x)$ is differentiable in $[-\pi, \pi]$:

$$\int_{-\pi}^{\pi} \cos nx \; u(x)\, dx = \underbrace{\frac{u(x) \sin nx}{n}\Big|_{-\pi}^{\pi}}_{0} - \underbrace{\frac{1}{n}\int_{-\pi}^{\pi} \sin nx \; u'(x)\, dx}_{\to 0 \text{ as } n \to \infty}$$

If u is not differentiable, approximate (integrable) u by smooth u.

Note also $\lim_{n\to\infty} \int_{-\pi}^{\pi} \sin nx \; u(x)\, dx = 0$

We want to show weak convergence $s_n(x) \longrightarrow \delta(x)$:

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} s_n(x)\, u(x)\, dx = u(0) \qquad u(x) \text{ smooth}$$

$$s_n(x) = \frac{1}{2\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}$$

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \frac{1}{2\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}\, u(x)\, dx = u(0)$$

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \frac{1}{2\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)} \left(u(x) - u(0)\right) dx = 0$$

$\left(\text{since } \int_{-\pi}^{\pi} s_n(x)\, dx = 1.\right)$

Now let $U(x) = \dfrac{u(x) - u(0)}{2\sin(x/2)}$.

l'Hôpital

Note $\lim\limits_{x \to 0} U(x) = \dfrac{u'(0)}{2(1/2)} = u'(0)$ exists

$$\lim_{x \to 0} U'(x) = \lim_{x \to 0} \frac{u'(x)\left(2\sin(x/2)\right) - \left(u(x) - u(0)\right)\cos(x/2)}{4\sin^2(x/2)}$$

$$= \lim_{x \to 0} \frac{u''(x)\left(2\sin(x/2)\right) + \cancel{u'(x)\cos(x/2)} - \cancel{u'(x)\cos(x/2)} + \cancel{(u(x) - u(0))}\frac{\sin(x/2)}{2}}{8\sin(x/2)\cos(x/2)}$$

$$= \frac{1}{4} u''(0)$$ also exists

We conclude that $U(x)$ is $C^1[-\pi, \pi]$.

Hence we can apply Riemann-Lebesgue to find

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} \sin\left(n + \frac{1}{2}\right)x \, U(x) \, dx = 0$$

We can actually prove that for $u(x)$ piecewise $C^1[-\pi, \pi]$:

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} s_n(x) u(x) \, dx = \frac{1}{2}\left(u(0^+) + u(0^-)\right).$$

(Same as convergence of Fourier series!)