Lecture 14: More on Laplace

Poisson integral formula:

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\,\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}\,d\phi$$

$$u(0,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\,d\phi$$

Average value

In fact this holds for any disk.

"local average"
see also finite diff

Theorem: $u(x,y)$ harmonic inside a disk of radius a centered at (x_0,y_0). Integrable boundary values.

$$u(x_0,y_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x_0 + a\cos\theta,\ y_0 + a\sin\theta)\,d\theta$$

proof: Laplace equation invariant under translation and scaling:

$$U(x,y) = u(x_0 + ax,\ y_0 + ay)$$

$$\nabla^2 U = a^2\nabla^2 u = 0, \qquad x^2+y^2<1.$$

Strong maximum principle: u nonconstant harmonic on bounded domain Ω, continuous on $\partial\Omega$.

Then u achieves max / min on $\partial\Omega$:

$$m = \min\{u(x,y) \mid (x,y) \in \partial\Omega\}$$

$$M = \max\{\quad " \quad\}$$

$$m < u(x,y) < M, \quad (x,y) \in \Omega.$$

proof: Let $M^* \geq M$ be max of u on $\Omega \cup \partial\Omega$.

Assume $u(x_0,y_0) = M^*$ at some interior (x_0,y_0).

$$u(x_0,y_0) = \frac{1}{2\pi}\oint_C u\,ds$$ Average on any circle centered on (x_0,y_0)

u is continuous (classical solution) and $\leq M^*$ on C, so its average is $< M^*$, unless $u = M^*$ on all of C.

But this is for any circle of and size, so by tiling with circles (Ω connected) conclude $u(x,y) = M^*$ in all of Ω. Contradiction!

(Same argument for min, or use $-u$.)

Uniqueness theorem for Dirichlet problem:

$u, \tilde{u}$ both solve $-\Delta u = f$ within bounded Ω, $u = \tilde{u}$ on $\partial\Omega$,

Then $u = \tilde{u}$ in all of Ω.

proof: $v = u - \tilde{u}$ satisfies $\Delta v = 0$ with $v = 0$ on $\partial\Omega$.
Hence $v = 0$ in all of Ω, by max/min principle.

Finally, note that any harmonic function is analytic at every point in Ω.

For disk of radius 1, Poisson integral formula:

$$u(r,\theta) = \frac{1}{\pi} \operatorname{Re} \int_{-\pi}^{\pi} h(\phi) \frac{z}{2(1-z)} \, d\phi \qquad z = re^{i(\theta-\phi)}$$

(Analytic for $|z| < 1$.)

Then use $U(x,y) = u(x_0 + ax, y_0 + ay)$ to scale to any disk inside Ω.