Lecture 13: Laplace equation

Laplace's equation. $u_{xx} + u_{yy} = 0$ in Ω

u is a **Harmonic function.**

(Poisson: $u_{xx} + u_{yy} = -f(x,y)$)

boundary

Dirichlet BCs: $u(x,y) = h(x,y)$ on $\partial\Omega$

Physically: $u(x,y)$ is a minimal surface (soap film) (small u)
is potential in cavity
is steady temperature distribution

Other BCs as for heat eq'n can also arise.

$u(x,y) = v(x)\,w(y)$ gives

$$v'' - \lambda v = 0, \qquad w'' + \lambda w = 0$$

Take $\lambda = -\omega^2 < 0$. Then $v \sim \cos \omega x,\ \sin \omega x$

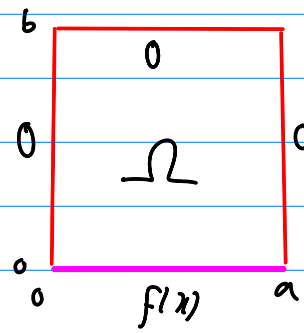
$w \sim e^{\pm \omega y}$

example: Ω = rectangle $[0,a] \times [0,b]$

$u(x,0) = f(x)$

All other edges 0

$u(x,b) = u(0,y) = u(a,y) = 0$



In easy cases we can "guess" at eigensolutions. $\lambda = -w^2$

$$u_n(x,y) = (A \cos wx + B \sin wx)(C \cosh wy + D \sinh wy)$$

$$u_n(0,y) = A \left(Ce^{wy} + De^{-wy} \right) = 0 \Rightarrow A = 0$$

$$u_n(a,y) = B \sin(wa) \left(y \right) = 0 \Rightarrow wa = n\pi, \quad n \in \mathbb{Z}.$$

$$u_n(x,b) = B \sin wx \left(C \cosh wb + D \sinh wb \right) = 0$$

So let $C = \sinh wb$

$D = -\cosh wb$

$$u_n(x,y) = B \sin wx \left(\sinh wb \cosh wy - \cosh wb \sinh wy \right)$$

$$= B \sin wx \ \sinh w(b-y)$$

$$\boxed{u_n(x,y) = \sin w_n x \ \sinh w_n (b-y), \quad w_n = \frac{n\pi}{a}}$$

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin w_n x \ \sinh w_n (b-y)$$

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin w_n x \ \sinh w_n b = f(x).$$

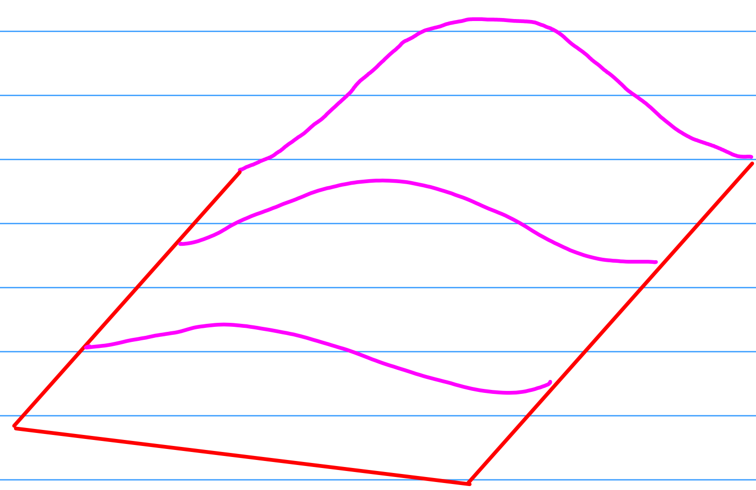
So if $b_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$, then

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} \frac{b_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi(b-y)}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}}$$

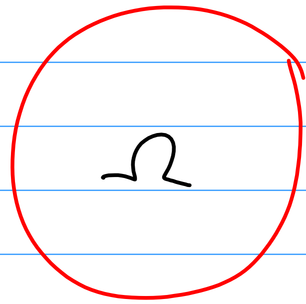
This solves the PDE since, under a mild condition like $\int_0^a |f(x)|\,dx < \infty$, we have $|b_n| \le M$.

Then $b_n \dfrac{\sinh\left(\frac{n\pi(b-y)}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)} \sim b_n e^{-\frac{n\pi y}{a}} \lesssim M e^{-n\pi y/a}$

These $\to 0$ exponentially fast for $0 < y \le b$.



Drum:



$$u(r,\theta)$$
$$u(1,\theta) = h(\theta)$$
$$u(r,\theta) = u(r,\theta+2\pi)$$
$$|u(r,\theta)| < M$$

Polar: $$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Separate variables again: $u(r,\theta) = v(r)w(\theta)$

$$\frac{r^2 v'' + r v'}{v} = -\frac{w''}{w} = \lambda$$

$$r^2 v'' + r v' - \lambda v = 0, \qquad w'' + \lambda w = 0$$

$\lambda > 0$ for periodic sol'ns

$$w(\theta) = \begin{cases} \cos n\theta \\ \sin n\theta \end{cases}, \quad \lambda_n = n^2 \qquad n \in \mathbb{Z}_{\geq 0}$$

The v eq'n is an Euler ODE: $v(r) = r^k$

Find solutions for $k = \pm n$.

$$v(r) = \begin{cases} r^n \\ r^{-n} \end{cases} \quad n \in \mathbb{Z}_{>0}$$

$$= \begin{cases} 1 \\ \log r \end{cases} \quad n = 0$$

General solution is thus

$$u(r,\theta) = \frac{a_0}{2} + b_0 \log r + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(a_n r^n \cos n\theta + b_n r^n \sin n\theta \right)$$

Boundedness at $r = 0$ means $a_n = b_n = 0$, $n < 0$.
$b_0 = 0$

$$u(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\theta + b_n \sin n\theta \right)$$
$$= h(\theta)$$

Hence, $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} h(\theta) \cos n\theta \, d\theta$, $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} h(\theta) \sin n\theta \, d\theta$.

Note: $z^n = r^n e^{in\theta} = r^n \cos n\theta + i r^n \sin n\theta$

Hence, $\mathrm{Re}\, z^n$ and $\mathrm{Im}\, z^n$ solve Laplace's equation

$z^2 = (x+iy)^2 = (x^2 - y^2) + 2i\, xy$ etc.

These are the harmonic polynomials.

Cute problem: $h(\theta) = \theta$

$-\pi < \theta < \pi$

$$h(\theta) \sim 2\left(\sin\theta - \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta - \cdots\right)$$

$$u(r,\theta) = 2\left(r\sin\theta - \frac{r^2}{2}\sin 2\theta + \frac{r^3}{3}\sin 3\theta - \cdots\right)$$

$$= 2\,\mathrm{Im}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{n}$$

$$= 2\,\mathrm{Im}\log(1+z)$$

$$= 2\psi, \qquad \psi = \tan^{-1}\frac{y}{1+x}.$$

Check: for $x = \cos\theta$, $y = \sin\theta$,

$$\tan^{-1}\frac{\sin\theta}{1+\cos\theta} = \tan^{-1}\frac{2\sin\theta/2\,\cos\theta/2}{1+(\cos^2\theta/2 - \sin^2\theta/2)} = \tan^{-1}(\tan(\theta/2)) = \frac{\theta}{2}$$

In fact, can sum general series:

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\underbrace{\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}}_{\text{Poisson kernel}}\,d\phi \qquad \begin{matrix} 0 \le r < 1 \\ -\pi < \theta < \pi \end{matrix}$$