Lecture 12: Separation of wave equation

Back to wave eq'n: $u_{tt} = c^2 u_{xx}$

had $e^{-\lambda t}$ before

Separation of variables Ansatz: $u(t,x) = w(t)\, v(x)$

Plug in, separate: $\dfrac{w''(t)}{w(t)} = c^2 \dfrac{v''(x)}{v(x)} = \lambda = \text{const}$

$$w'' - \lambda w = 0, \qquad v'' - \frac{\lambda}{c^2} v = 0$$

Solve as before:

$\lambda > 0$ $\sin, \cos$

$\lambda < 0$ $e^{\pm}$ or $\sinh, \cosh$

$\lambda = 0$ $1, x$

Assume solution is bounded in $\mathbb{R}^2$: $0 < \lambda = \omega^2$

$$u(t,x) = \sum_n \left(A_n \cos \omega_n t + B_n \sin \omega_n t \right)\left(C_n \cos\left(\frac{\omega_n x}{c}\right) + D_n \sin\left(\frac{\omega_n x}{c}\right)\right)$$

$$\cos a \cos b = \frac{1}{2}\left(\cos(a+b) + \cos(a-b) \right)$$

$$\sin a \sin b = \frac{1}{2}\left(\cos(a+b) - \cos(a-b) \right)$$

$$\cos a \sin b = \frac{1}{2}\left(\sin(a+b) - \sin(a-b) \right)$$

$$\sin a \cos b = \frac{1}{2}\left(\sin(a+b) + \sin(a-b) \right)$$

By using these we have

$$u(t,x) = \sum_n \left[A_n^{(R)} \cos\left(\frac{w_n}{c}(x - ct)\right) + B_n^{(R)} \sin\left(\frac{w_n}{c}(x-ct)\right)\right]$$

$$+ \sum_n \left[A_n^{(L)} \cos\left(\frac{w_n}{c}(x+ct)\right) + B_n^{(R)} \sin\left(\frac{w_n}{c}(x+ct)\right)\right]$$

Recall the earlier general solution: $u(t,x) = p(x-ct) + q(x+ct)$

This is the same but in Fourier form. (Technically p, q should be periodic, but can let period $\to \infty$ and use Fourier transform.)

Recall now d'Alembert's formula:

$$u(t,x) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\, dz$$

solves $u_{tt} = c^2 u_{xx}$, $u(0,x) = f(x)$, $u_t(0,x) = g(x)$.

To solve Dirichlet problem $u(t,0) = u(t,l) = 0$ on $0 < x < l$, use either Fourier series as above ($\sin$ in x), or use odd periodic extension:

$$\tilde{f}(x) = -\tilde{f}(-x), \quad \tilde{g}(x) = -\tilde{g}(-x),$$

$$\tilde{f}(x+2l) = \tilde{f}(x), \quad \tilde{g}(x+2l) = \tilde{g}(x),$$

Then

$$u(t,0) = \frac{1}{2}\underbrace{\left(\tilde{f}(-ct) + \tilde{f}(ct)\right)}_{0\,!} + \frac{1}{2c}\overbrace{\int_{-ct}^{ct} \tilde{g}(z)\,dz}^{0\,!} = 0$$

$$u(t,\ell) = \frac{1}{2}\left(\tilde{f}(\ell-ct) + \tilde{f}(\ell+ct)\right) + \frac{1}{2c}\int_{\ell-ct}^{\ell+ct} \tilde{g}(z)\,dz$$

$\downarrow$ periodic

$$= \frac{1}{2}\left(\tilde{f}(-\ell-ct) + \tilde{f}(\ell+ct)\right) + \frac{1}{2c}\int_{-ct}^{ct} \underbrace{\tilde{g}(z'+\ell)}_{-\tilde{g}(-\tilde{z}'+\ell)\ \text{odd, in } \tilde{z}'}\,dz'$$

$$= 0$$

$$\left[\tilde{g}(x+\ell) = \tilde{g}(x-\ell) = -\tilde{g}(-x+\ell)\right]$$

So we use the full d'Alembert solution with $\tilde{f}, \tilde{g}$, but only evaluate the result on $0 < x < \ell$. "Window" on periodic wave. "interference" picture

Similar trick for Neumann ($u_x = 0$) conditions using even periodic extension.

Q: Can this work for $u(t,0)=0$, $u_x(t,\ell)=0$?