Lecture 11: More on heat equation

A quick look at inhomogeneous B.C.:

$$u_t = \gamma u_{xx}, \quad u(t,0) = \alpha, \quad u(t,\ell) = \beta \quad t \geq 0$$

The general method (for time-independent B.C.) is to solve the steady problem: (equilibrium)

$$0 = \gamma U''(x), \quad U(0) = \alpha, \quad U(\ell) = \beta$$

$$\Rightarrow U(x) = \alpha + \frac{\beta - \alpha}{\ell} x$$

Then define $\tilde{u}(t,x) = u(t,x) - U(x)$.

Easy to see that $\tilde{u}$ has homogeneous B.C.s.

Hence, can re-use homogeneous sol'n from last time:

$$u(t,x) = \alpha + \underbrace{\frac{\beta - \alpha}{\ell} x}_{} + \sum_{n=1}^{\infty} \tilde{b}_n e^{-\gamma n^2 \pi^2 t/\ell^2} \sin\left(\frac{n\pi x}{\ell}\right).$$

can substitute Fourier series to equate coeffs of $f(x)$.

When BCs are time-dep., find particular solution instead!
Get "source term" in $\tilde{u}$ equation.

Robin BCs: Things get hairier with more general BCs:

$$u_t = u_{xx}, \quad u(t,0)=0, \quad u_x(t,1) + \beta u(t,1) = 0.$$

$\beta = 0$ is easy, so assume $\beta \neq 0$.

$u = e^{-\lambda t} v(x)$, etc... (separation of variables again)

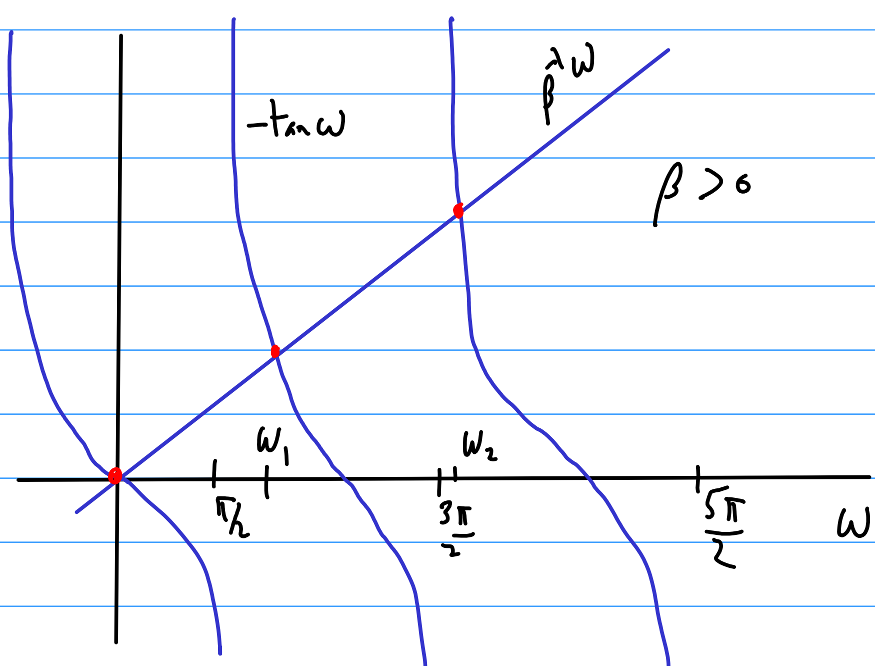
$$v'' + \lambda v = 0, \quad v(0) = 0, \quad v'(1) + \beta v(1) = 0.$$

Again let $w = \sqrt{\lambda}$.

$$v(x) = \underset{0}{\cancel{a}} \cos wx + \underset{1}{b} \sin wx$$

$$v'(1) + \beta v(1) = w \cos w + \beta \sin w = 0.$$

transcendental eq'n: $\beta^{-1} w = -\tan w$ $\qquad$ $w=0$ trivial solution



∞ number of solutions,

$$w_n = \frac{(2n-1)}{2}\frac{\pi}{2}$$

as $n \to \infty$.

Also works for $\beta < 0$

Here $\lambda > 0$ so these are exp. decaying solutions.

$$u_n(t,x) = e^{-\lambda_n t} \sin(\omega_n x), \qquad \omega_n = \sqrt{\lambda_n}$$

($\lambda = 0$ is a special case: $v(x) = \underset{0}{\cancel{a}} + \overset{1}{b}x$

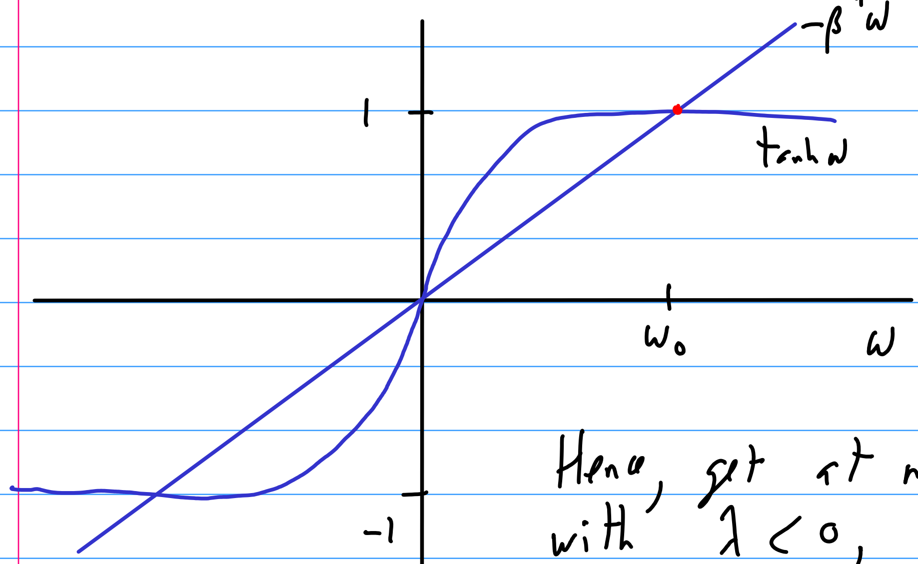
$$v'(1) + \beta v(1) = 1 + \beta = 0 \Rightarrow \beta = -1$$

Steady solution only exists if fluxes are balanced.)

Now suppose $\lambda = -w^2 = 0 \Rightarrow v(x) = \sinh wx$

$$v'(1) + \beta v(1) = w \cosh w + \beta \sinh w = 0$$

$$-\beta^{-1} w = \tanh w$$



Get trivial solution unless

$0 < -\beta^{-1} < 1$

or

$-\infty < \beta < -1$

Hence, get at most one eigensolution with $\lambda < 0$, when $\beta < -1$.

All other eigensolutions have $\lambda > 0$. (Decaying)

$$u_0(t,x) = e^{\lambda_0 t} \sinh \omega_0 x$$ Growing eigenmode!

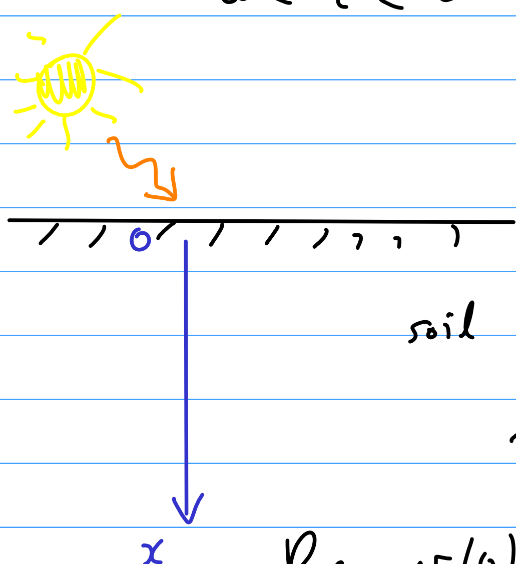
This corresponds to a net heat flux into the system.

Another type of problem: ground heating

$u_t = \gamma u_{xx}$, $\quad x > 0$, $\quad u(t,0) = a \cos \omega t$

$-\infty < t < \infty$ $\qquad |u(t,x)| < M$ bounded

$u(t,\infty) = 0$



Bounded in time mean

$$u(t,x) = e^{i\omega t} v(x)$$

$$\gamma v'' = i\omega v$$

$$\operatorname{Re} v(0) = a, \qquad v(\infty) = 0.$$

Put $v(x) = e^{\alpha x}$: $\quad \gamma \alpha^2 = i\omega$

$$\alpha = \pm\sqrt{i\omega/\gamma} = \pm(1+i)\sqrt{\frac{\omega}{2\gamma}}$$

Take "$-$" to satisfy $v(\infty) = 0$.

$$v(x) = v_0 \, e^{-(1+i)\sqrt{\frac{w}{2r}}\,x}$$

$$= v_0 \, e^{-\sqrt{\frac{w}{2r}}\,x} \left(\cos \sqrt{\frac{w}{2r}}\,x + i \sin \sqrt{\frac{w}{2r}}\,x \right)$$

$\operatorname{Re} v(0) = a \Rightarrow v_0 = a.$

So finally:

$$u(t,x) = \operatorname{Re}\left[e^{iwt} \, a \, e^{-(1+i)\sqrt{\frac{w}{2r}}\,x} \right]$$

$$\boxed{u(t,x) = a \, e^{-\sqrt{\frac{w}{2r}}\,x} \cos\left(\sqrt{\frac{w}{2r}}\,x - wt \right)}$$

$$\sqrt{\frac{w}{2r}}\,x - wt = \sqrt{\frac{w}{2r}}\left(x - \underbrace{\sqrt{2rw}}_{\text{speed of propagation}}\, t \right)$$

$\sqrt{\frac{2r}{w}}$ is a "skin depth". Shallower for w large.