Lecture 10: Heat equation

Heat equation: $$\frac{\partial \varepsilon}{\partial t} + \frac{\partial w}{\partial x} = 0$$

$\varepsilon = \sigma(x) u(t,x)$ energy density

σ: density × heat capacity

κ: thermal conductivity

$w = -\kappa(x) \dfrac{\partial u}{\partial x}$ Fourier's law of cooling (energy flux)

Heat Equation

$$\frac{\partial}{\partial t}(\sigma u) = \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial u}{\partial x}\right) + h(t,x)$$

h: internal source

$a < x < b$

$u(t_0, x) = f(x), \quad a < x < b$

Robin boundary condition: $u_x(t,a) + \beta(t) u(t,a) = \tau(t)$.

$u(t,a) = \alpha(t)$ temperature specified (Dirichlet)

$u_x(t,a) = \tau(t)$ flux specified (Neumann)
($\tau = 0 \Rightarrow$ insulated)

Also: periodic.

Really we will mostly deal with case with constants:

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2} \qquad \gamma = \frac{\kappa}{\sigma}$$

Simple problem: bar (1D) of length ℓ, zero temp at ends

$$u(t,0) = u(t,\ell) = 0, \quad t \geq 0$$

$$u(0,x) = f(x), \quad 0 \leq x \leq \ell$$

Put $u(t,x) = e^{-\lambda t} v(x)$ (anticipate sign of λ)

$$\gamma v'' + \lambda v = 0, \qquad v(0) = v(\ell) = 0.$$

Find $v(x) = a \cos \omega x + b \sin \omega x$, $\quad \omega = \sqrt{\frac{\lambda}{\gamma}}$.

B.C. at $x = 0 \Rightarrow a = 0$

B.C. at $x = \ell \Rightarrow \omega \ell = n\pi, \quad 0 < n \in \mathbb{Z}$.

Eigensolutions are: $u_n(t,x) = \exp\left(-\frac{\gamma n^2 \pi^2 t}{\ell^2}\right) \sin\left(\frac{n\pi x}{\ell}\right)$.

Assemble into series:

$$u(t,x) = \sum_{n=1}^{\infty} b_n u_n(t,x) = \sum_{n=1}^{\infty} b_n e^{-\gamma n^2 \pi^2 t/\ell^2} \sin\left(\frac{n\pi x}{\ell}\right)$$

$$u(0,x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right) = f(x).$$

Fourier sine series on $[0, \ell]$. This is because solution is not periodic, but rather has 0 endpoints. Automatically satisfied by periodic odd function on $[-\ell, \ell]$.

$$b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx, \quad n > 0$$

For example, consider an isothermal initial condition $f(x) = f_0$, $0 < x < \pi$. Does not satisfy BC but ok: like making contact with a hot heating element.

Then $b_n = \dfrac{2}{\ell} f_0 \displaystyle\int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) dx$

$$= \frac{2f_0}{\ell} \frac{\ell}{n\pi} \cos\left(\frac{n\pi x}{\ell}\right)\Big|_{\ell}^{0} = \frac{2f_0}{n\pi}\left(1-(-1)^n\right).$$

$$u(t,x) = \frac{4f_0}{\pi} \sum_{n \text{ odd}} \frac{e^{-\gamma n^2\pi^2 t/\ell^2}}{n} \sin\left(\frac{n\pi x}{\ell}\right).$$

A few remarks:

- If $f(x)$ is integrable, it has bounded Fourier coeffs:

$$|b_n| \le \frac{2}{\ell}\int_0^\ell \left| f(x)\sin\left(\frac{n\pi x}{\ell}\right)\right| dx \le \frac{2}{\ell}\int_0^\ell |f(x)|dx = M$$

This property holds even for noisy initial data.

But then $\left| b_n e^{-\gamma n^2\pi^2 t/\ell^2} \sin\left(\frac{n\pi x}{\ell}\right)\right| \le M e^{-\gamma n^2\pi^2 t/\ell^2}$

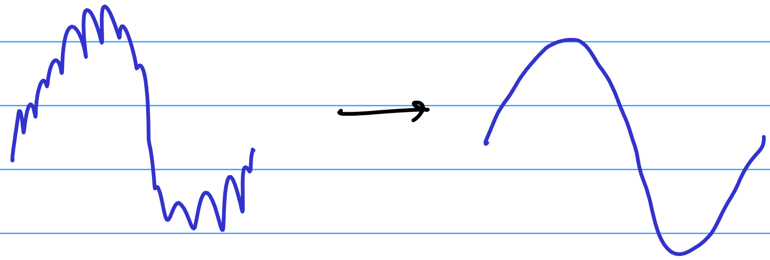
Hence $u(t,x) \to 0$ exponentially fast (uniform temp.)

- For large t, keep only the slowest-decaying exponential, typically $n=1$ (when $b_1 \neq 0$).

Then $u(t,x) \sim \frac{4f_0}{\pi} e^{-\gamma\pi^2 t/\ell^2} \sin\left(\frac{\pi x}{\ell}\right)$, $t \to \infty$

So ultimately almost any initial condition settles into the $\sin\left(\frac{\pi x}{\ell}\right)$ eigenfunction.

- Initial fluctuations disappear rapidly.
This can be used for denoising a signal



For t not too large!

- The Fourier coeffs decay exponentially with n for $t>0$. So solution is C^∞ even if $f(x)$ isn't.

- This also implies the backward heat equation

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^2 u}{\partial x^2}$$

is <u>ill-posed</u> for $t>0$.

$\notin C^\infty$

Indeed, assume we evolve an integrable $f(x)$ backward in time to $t=-t_0$, $t_0>0$.

But then evolving this back forward from $-t_0$ to 0 should recover $f(x)$. But this would imply $f \in C^\infty$, which we assumed it didn't.

The problem is that high-frequency noise grows as $e^{\gamma n^2\pi^2 t/\ell^2}$ when evolved backward! So there is no convergence.

- In practice, what if you actually need to solve the eq'n backwards? (i.e., time of death on CSI!)

OK, but truncate Fourier series.

Can also be used in <u>image sharpening</u>