Lecture 9: Convergence of Fourier series

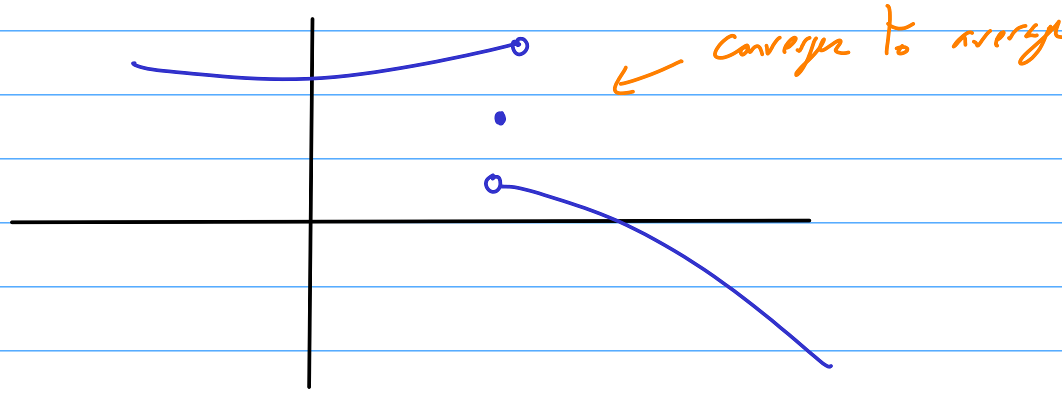
Now that we've define the appropriate class of functions, can state the main convergence theorem:

Theorem: Let $\tilde{f}(x)$ be 2π-periodic, piecewise C^1.

For any $x \in \mathbb{R}$, Fourier series converges to

$$\frac{1}{2}\left[\tilde{f}(x^+) + \tilde{f}(x^-)\right].$$

implies $\tilde{f}(x)$ if continuous



This is pointwise convergence. It is not as strong as uniform convergence.

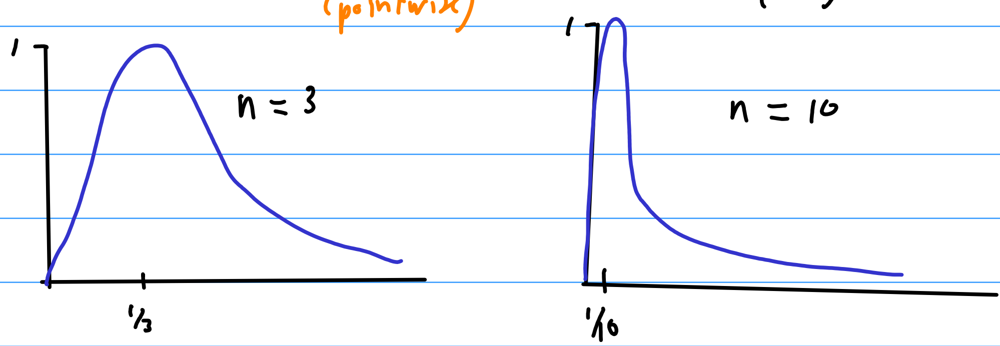
This means that for each point x the series eventually converges, but the rate of convergence may depend on x

A simple example of this is the sequence

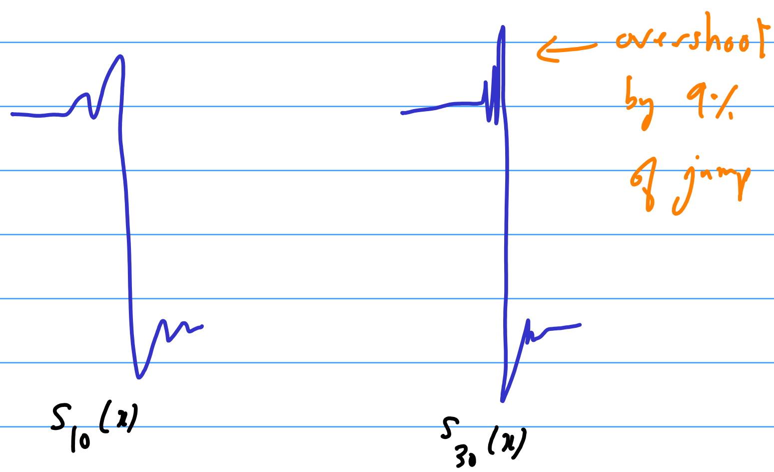
$$v_n(x) = \frac{2nx}{1+n^2x^2}.$$

(Not a Fourier series, just motivating example.)

We have $\lim_{n\to\infty} v_n(x) = 0$. (pointwise) But $v_n\left(\frac{1}{n}\right) = 1$.



For Fourier series, this leads to Gibbs phenomenon at discontinuities:



The overshoot does not go away as $n \to \infty$. But it gets squeezed against the jump point.

For an even function get Fourier cosine series
odd function get Fourier sine series.

We can "force" a sine or cosine series by considering the even / odd extension of $f(x)$, $0 < x < \pi$.

For example: take $f(x) = \sin x$ extended to even function.

$$a_k = \frac{2}{\pi} \int_0^{\pi} \sin x \cos kx \, dx = \begin{cases} 2/\pi, & k = 0 \\ 0, & k \text{ odd} \\ -\frac{4}{(k^2-1)\pi}, & 0 < k \text{ even} \end{cases}$$

This means

$$|\sin x| \sim \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos 2kx}{4k^2 - 1}$$

Complex form: often much more convenient.

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} \, dx$$

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

$$c_k = \langle f, e^{ikx} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} \, dx$$

We get uniform convergence of Fourier series if

$$\sum_{k=-\infty}^{\infty} |c_k| < \infty .$$

(This follows from Weierstrass M-test.

$$\left| \sum_{k=-n}^{n} c_k e^{ikx} \right| \leq \sum_{k=-n}^{n} |c_k| .)$$

This means that if we have $|c_k| \sim \frac{1}{|k|^\alpha}$,

need $\alpha > 1$ for uniform convergence.

Uniform convergence gives C^0 function

(Unif. convergent limit of C^0 functions.)

But since $f'(k) \sim \sum ik c_k e^{ikx}$,

this will be unif. conv. if $|k c_k| \sim \frac{1}{|k|^\alpha}$, $\alpha > 1$

so $|c_k| \sim \frac{1}{|k|^{\alpha+1}}$ $\alpha + 1 > 2$

So Fourier coeffs of C^1 function decay faster than $\frac{1}{|k|^2}$

Theorem: Let $0 \le n \in \mathbb{Z}$. If Fourier coeffs of $f(x)$ satisfy

$$\sum_{k=-\infty}^{\infty} |k|^n |c_k| < \infty$$

then the Fourier series converges unif. to $\tilde{f}(x) \in C^n$.

If series converges faster than any power, then C^∞.

example:

$$\sum_{k=-\infty}^{\infty} e^{-|k|x} e^{ikx}$$