Lecture 8: Fourier series

Goal: Represent $f(x)$ as convergent series:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos kx + b_k \sin kx \right)$$

Inner product: $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x)\, dx$.

$$\|f\|^2 = \sqrt{\langle f, f \rangle}$$

Orthogonality:

$\langle \cos kx, \cos lx \rangle = \langle \sin kx, \sin lx \rangle = 0$, $k \neq l$

$\langle \cos kx \sin lx \rangle = 0$

$\| 1 \| = \sqrt{2}$, $\| \cos kx \| = \| \sin kx \| = 1$.

Given these, and not worrying about convergence, we have (if series converges to $f(x)$)

$a_l = \langle f, \cos lx \rangle$, $l \geq 0$ $\quad b_l = \langle f, \sin lx \rangle$ $l > 0$

example: $f(x) = x \quad , \quad -\pi < x < \pi$

$$x \sim 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \dots\right)$$

Partial sums are called trigonometric polynomials:

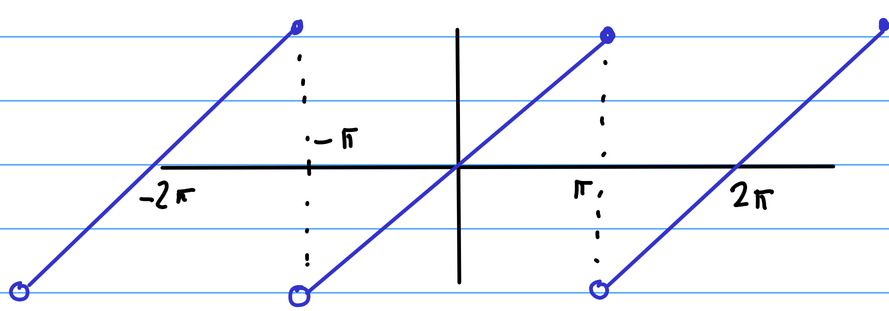
$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k \cos kx + b_k \sin kx\right)$$

Convergence means: $\lim_{n\to\infty} s_n(x) = \tilde{f}(x)$ ← not necessarily $f(x)$!

Let $f(x)$, $-\pi < x \leq \pi$. 2π-periodic extension of f is unique function $\tilde{f}(x)$ that satisfies

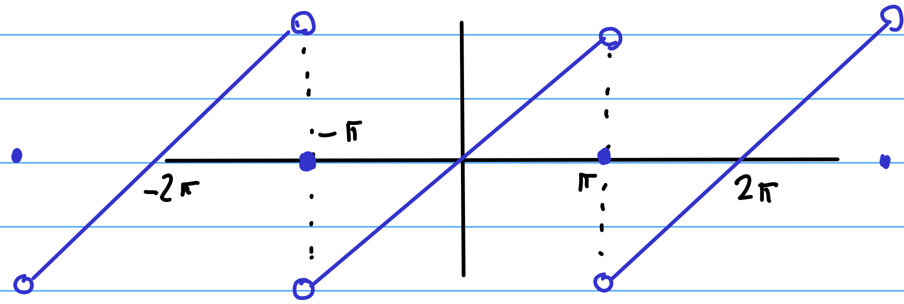
$$\tilde{f}(x + 2\pi m) = f(x) \quad , \text{ for some } m(x) \in \mathbb{Z}.$$

example: 2π-periodic extension of x is sawtooth.



It will turn out to be a better choice to set the function equal to the average at discontinuities:

$$\tilde{f}(\pi) = \tilde{f}(-\pi) = \frac{1}{2}\left[f(\pi) + f(-\pi)\right]$$



With that choice, Fourier series converge everywhere to $\tilde{f}(x)$:

$$2\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin(kx)}{k} = \begin{cases} x, & -\pi < x < \pi \\ 0, & x = \pm\pi \end{cases}$$

(Left-hand side is 2π-periodic, by construction.)

Discontinuities in $f(x)$ and its derivatives are the central aspect that affects convergence. We thus define:

$f(x)$ is piecewise continuous on $[a, b]$ if it is continuous except possibly at finite # of points

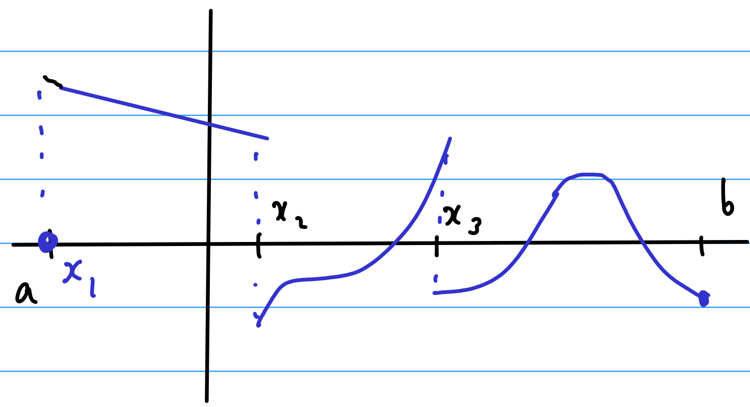
$$a \leq x_1 < x_2 < \cdots < x_n \leq b.$$

Furthermore, require that left/right limits exist:

$$f(x_k^-) = \lim_{x \to x_k^-} f(x), \quad f(x_k^+) = \lim_{x \to x_k^+} f(x)$$

for $k = 1, \ldots, n$. (One-sided limit at endpoints)

Note that nothing is required of $f(x_k)$ (could be undefined)



Magnitude of jumps $\beta_k = f(x_k^+) - f(x_k^-)$

(x_k is removable if $\beta_k = 0$.)

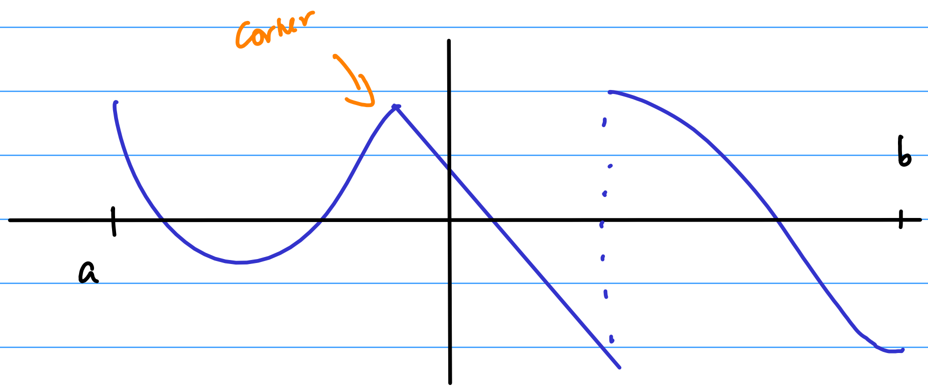
Simplest example: step function $\sigma(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$

$\sigma(0^+) - \sigma(0^-) = 1.$

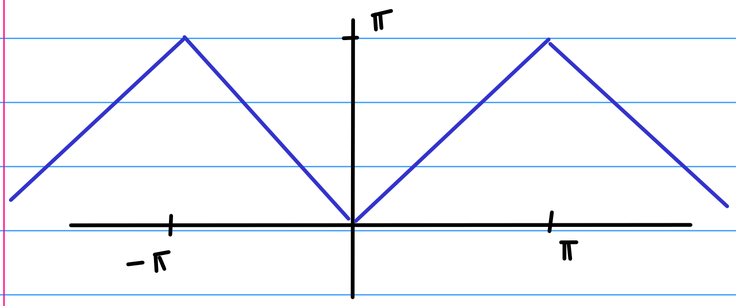
Now a stronger version:

$f(x)$ is piecewise C^1 on $[a,b]$ if it is continuously differentiable except at a finite # of points.

As for piecewise C^0 with in addition $f'(x^\pm)$ existing.



example: $f(x) = |x|$ is piecewise C^1.



Can generalize to piecewise C^n functions.