Lecture 7: Eigensolutions

Linear evolution equations: $\dfrac{\partial u}{\partial t} = \mathcal{L}[u]$

$\mathcal{L}[u]$ = linear differential operator
(only space derivatives)

examples: $\mathcal{L}[u] = \dfrac{\partial^2 u}{\partial x^2}$ heat equation

$= -c(x)\dfrac{\partial u}{\partial x}$ transport equation

$\mathcal{L}[u+v] = \mathcal{L}u + \mathcal{L}v, \quad \mathcal{L}[c(t)u] = c(t)\mathcal{L}u$

Motivation: $\underline{u}(t) \in \mathbb{R}^n$

$\dfrac{d\underline{u}}{dt} = A\underline{u}$ $\quad$ A $n \times n$ const. matrix

Solve by putting $\underline{u}(t) = e^{\lambda t}\underline{v}$

$\Rightarrow \quad \lambda \underline{v} = A\underline{v}$, so $\underline{v}$ is eigenvector of A
λ eigenvalue

For different eigenvals/vecs, write $u_k(t) = e^{\lambda_k t}\underline{v}_k$.

The general solution is then

$$\underline{u}(t) = c_1 e^{\lambda_1 t} \underline{v}_1 + \cdots + c_n e^{\lambda_n t} \underline{v}_n$$

assuming n linearly indep. eigenvecs $\underline{v}_k$. complete system

(Otherwise, use Jordan form, with eigenfunctions like $te^{\lambda t}$.)

Note that $\lambda_k, \underline{v}_k$ may be complex, but must come in conjugate pairs. Adjust c_k's to keep things real.

Now back to evolution PDEs: write $u(t,x) = e^{\lambda t} v(x)$.

$$\frac{\partial u}{\partial t} = \mathcal{L}[u] \quad \Rightarrow \quad \lambda v = \mathcal{L}[v].$$ eigenequation

However, will require an ∞ number of λ, $v(x)$.

Take $\mathcal{L} = \frac{\partial^2}{\partial x^2}$. Then $v'' = \lambda v$.

Solutions come in several forms, depending on $\lambda > 0, 0, < 0$, or even complex.

For instance, $\lambda = -\omega^2 < 0$ leads to

$$u(t,x) = e^{-\omega^2 t} \left(c_1 \cos \omega x + c_2 \sin \omega x \right).$$

The exact nature of allowable λ's is determined by boundary conditions and boundedness requirements

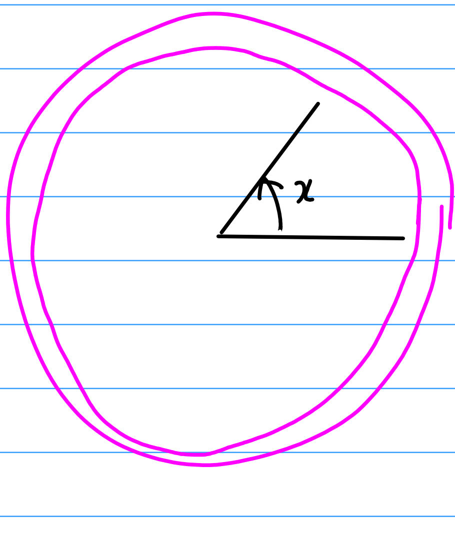
(So for heat equation $\lambda = +\omega^2 > 0$ is not allowed, since solutions then blow up as $t \to \infty$.)

Finite linear combinations are solutions.

i.e. $u(t,x) = c_1 e^{-t} \cos x + c_2 e^{-4t} \sin 2x + c_3 x + c_4$.

Infinite series are of course trickier, but of paramount importance.

<u>Heated ring:</u> As a motivating example, consider



$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Thin ring.
x = angle.

$$\left.\begin{aligned} u(t,-\pi) &= u(t,\pi) \\ u_x(t,-\pi) &= u_x(t,\pi) \end{aligned}\right\} \text{periodic}$$

$$u(0,x) = f(x)$$

Put $u(t,x) = e^{\lambda t} v(x)$

$$\lambda v = v''$$

$$v(-\pi) = v(\pi)$$

$$v'(-\pi) = v'(\pi)$$

all others also periodic! Why?

Find nonzero solutions (nontrivial)

Periodicity suggests we want sin/cos in x. $\lambda = -\omega^2$

$$v(x) = a \cos \omega x + b \sin \omega x$$

We can satisfy both periodicity requirements with

$\omega = k, \quad k = 0, 1, 2, 3, \cdots$ (No need to consider negative ω. Why?)

! careful

Conclude: $\lambda_k = -k^2, \quad v_k(x) = \cos kx$

$$\tilde{v}_k(x) = \sin kx$$

This gives $u_k(t,x) = e^{-k^2 t} \cos kx$

$$\tilde{u}_k(t,x) = e^{-k^2 t} \sin kx$$

Write full solution as a superposition: Fourier series

$$u(t,x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} e^{-k^2 t} \left[a_k \cos kx + b_k \sin kx \right]$$

$k=0$, annoying!

Why only one sol'n for $k=0$? The other is $x \Rightarrow$ not periodic

The a's and b's are called Fourier coefficients.
Prescribed by

$$f(x) = u(0,x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos kx + b_k \sin kx \right]$$

Some obvious questions:

- How to find a, b's? projection
- Does the series converge? How? To what?
- What kinds of $f(x)$ are allowed?
- Does the ∞ series solve the original eq'n?