Lecture 6: Wave equation

$u(t,x)$ = displacement

ρ = density

κ = stiffness/tension

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial u}{\partial x}\right)$$

ρ, κ const $\Rightarrow$ $$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \qquad c = \sqrt{\frac{\kappa}{\rho}}$$ wave speed

Need two initial conditions:

$$u(0,x) = f(x), \quad u_t(0,x) = g(x)$$

Write $$\Box u = (\partial_t^2 - c^2\partial_x^2)u = 0$$

wave operator, D'Alembertian

$$\Box = (\partial_t - c\,\partial_x)(\partial_t + c\,\partial_x)$$ for classical solutions

$\Rightarrow$ solutions to transport eq'n also solutions to wave eq'n,

but bidirectional

All solutions can be written

$$u(t,x) = p(\xi) + q(\eta) \qquad (*)$$

with $\xi = x - ct$, $\eta = x + ct$.

How do we show this?

Write $u(t,x) = v(x-ct,\ x+ct) = v(\xi,\eta)$

$$= u\left(\frac{\eta-\xi}{2c}, \frac{\eta+\xi}{2c}\right)$$

Then: $\Box u = -4c^2 \dfrac{\partial^2 v}{\partial\xi\,\partial\eta}$, from which $(*)$ follows.

Solve IVP: $u(0,x) = p(x) + q(x) = f(x)$

$$u_t(0,x) = -cp'(x) + cq'(x) = g(x)$$

$$\Rightarrow \begin{array}{l} p' + q' = f' \\ p' - q' = -\dfrac{g}{c} \end{array} \Rightarrow 2p' = f' - \frac{g}{c}$$

Hence: $$p(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_0^x g(z)\,dz + a$$

This gives $q(x) = f(x) - p(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_0^x g(z)\,dz - a$

So finally

$$u(t,x) = p(\xi) + q(\eta)$$

$$= \frac{1}{2}\left(f(\xi) + f(\eta)\right) - \frac{1}{2c}\int_0^{\xi} g(z)\,dz + \frac{1}{2c}\int_0^{\eta} g(z)\,dz$$

$$\boxed{u(t,x) = \frac{1}{2}\left(f(\xi) + f(\eta)\right) + \frac{1}{2c}\int_{\xi}^{\eta} g(z)\,dz}$$

This is d'Alembert's solution. It involves the superposition of left- and right-traveling waves.

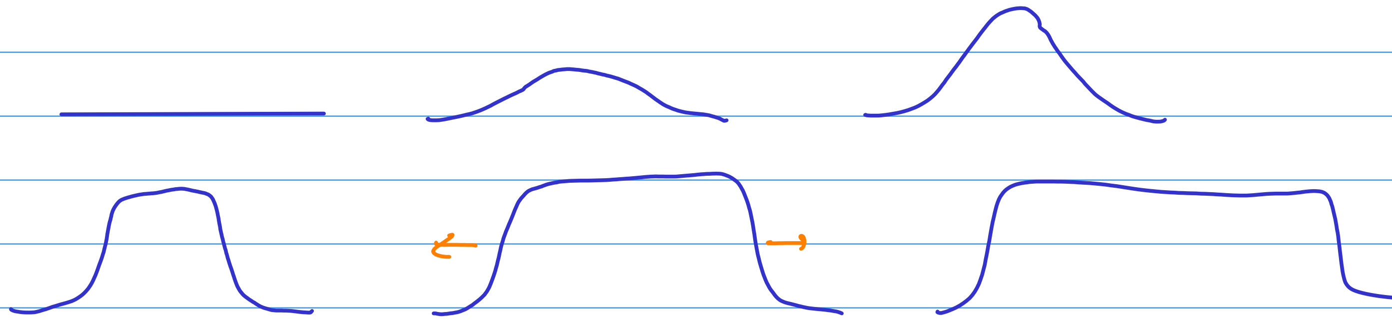
Sometimes not apparent;

$$u(t,x) = \cos(ct)\cos x = \frac{1}{2}\cos(x-ct) + \frac{1}{2}\cos(x+ct)$$

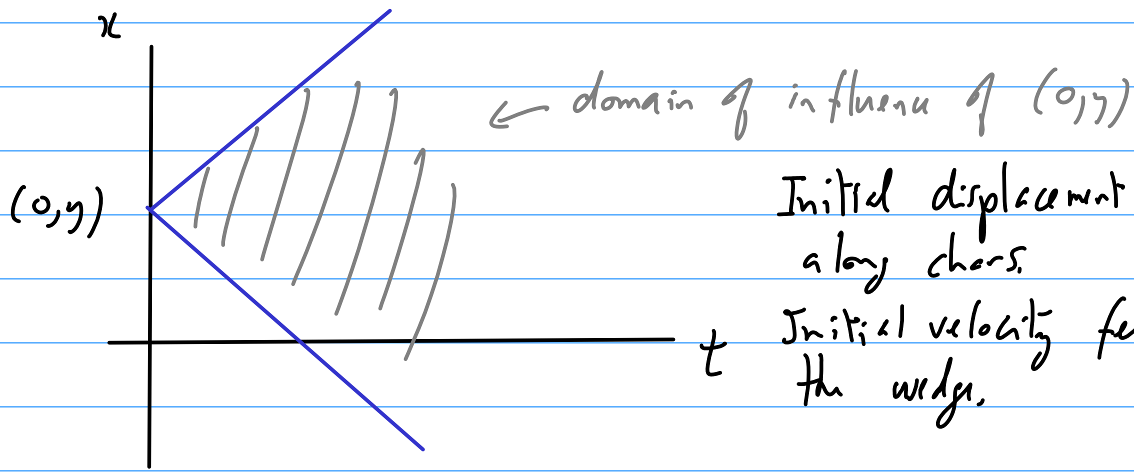
Also weird; $f(x) = 0$, $g(x) = e^{-x^2}$

$$u(t,x) = \frac{1}{2c}\int_{x-ct}^{x+ct} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{4c}\left[\operatorname{erf}(x+ct) - \operatorname{erf}(x-ct)\right]$$

$$\operatorname{erf} x := \frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}\,dx, \qquad \operatorname{erf}(\pm\infty) = \pm 1$$



Seems a bit counterintuitive at first, because there is no "reflection". Give finite impulse.



Initial displacement propagates along chars.

Initial velocity felt throughout the wedge.

<u>Forcing</u>: $\Box u = F(t, x)$.

$$-4c^2 \frac{\partial^2 v}{\partial \xi \partial \eta} = F\left(\frac{\eta - \xi}{2c}, \frac{\eta + \xi}{2c}\right)$$

$$\frac{\partial v}{\partial \xi}(\xi, \eta) - \frac{\partial v}{\partial \xi}(\xi, \xi) = -\frac{1}{4c^2} \int_{\xi}^{\eta} F\left(\frac{\zeta - \xi}{2c}, \frac{\zeta + \xi}{2c}\right) d\zeta$$

$$\frac{\partial v}{\partial \xi}(\xi,\xi) = \frac{1}{2c}\frac{\partial u}{\partial t}(0,\xi) + \frac{1}{2}\frac{\partial u}{\partial x}(0,\xi)$$

This vanishes if we take $u(0,x) = u_t(0,x) = 0$.

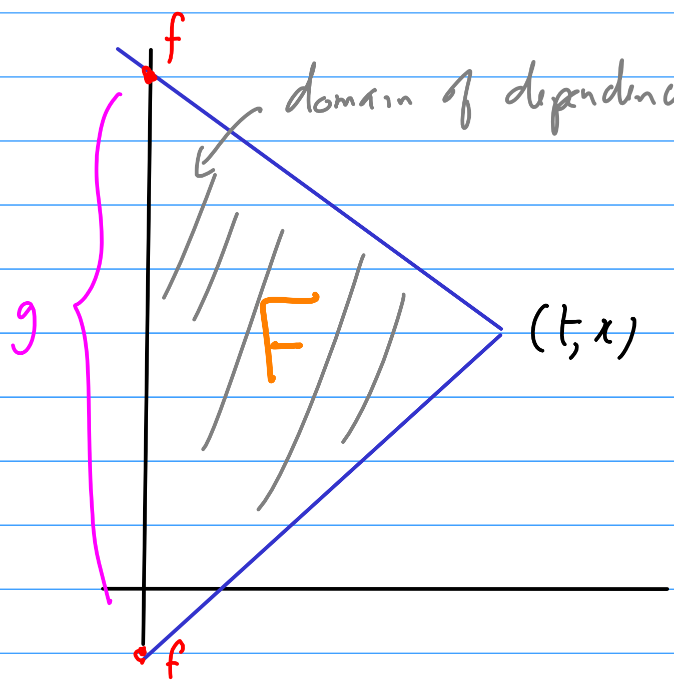
$$\frac{\partial v}{\partial \xi}(\xi,\eta) = -\frac{1}{4c^2}\int_\xi^\eta F\left(\frac{\zeta-\xi}{2}, \frac{\zeta+\xi}{2}\right) d\zeta$$

Now integrate w.r.t. to ξ on $\xi \leq \chi \leq \eta$:

$$\underbrace{v(\eta,\eta)}_{u(0,\eta)=0} - \underbrace{v(\xi,\eta)}_{u(t,x)} = \frac{-1}{4c^2}\int_\xi^\eta\int_\chi^\eta F\left(\frac{\zeta-\chi}{2}, \frac{\zeta+\chi}{2}\right) d\zeta\, d\chi$$

Go back to (t,x):

$$u(t,x) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} F(s,y)\, dy\, ds$$



example: $u_{tt} = u_{xx} + \sin(\omega t)\sin x$, $u(0,x) = 0$, $u_t(0,x) = 0$

$$u(t,x) = \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} \sin(\omega s)\sin y \, dy \, ds$$

$$= \frac{1}{2}\int_0^t \sin \omega s \left(\cos(x-t+s) - \cos(x+t-s)\right) ds$$

$$= \begin{cases} \dfrac{\sin \omega t - \omega \sin t}{1-\omega^2} \sin x, & 0 < \omega \neq 1 \\ \dfrac{\sin t - \underbrace{t}_{\text{resonance}} \cos t}{2} \sin x, & \omega = 1 \end{cases}$$