Lecture 5: Shocks

"Mass" in an interval: $M_{a,b}(t) = \int_a^b u(t,x)\,dx$

(*)

Assume $u(t,x)$ is a classical solution to $u_t + uu_x = 0$.

$$\frac{dM_{a,b}}{dt} = \int_a^b \frac{\partial u}{\partial t}\,dx = -\int_a^b \underbrace{uu_x}_{\frac{1}{2}(u^2)_x}\,dx$$

$$= -\left[\frac{1}{2}u^2\right]_{x=a}^{b} = \underbrace{\frac{1}{2}u^2(t,a) - \frac{1}{2}u^2(t,b)}_{\text{net mass flux through } [a,b]}$$

$M_{a,b}$ only changes through endpoints.

If $u(0,x) = f(x)$ has finite total mass: $\left|\int_{-\infty}^{\infty} f\,dx\right| < \infty$,

then $f(x)$ must decay at $\pm\infty$.

Conclude: $$\int_{-\infty}^{\infty} u(t,x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$$

Mass is thus conserved until formation of discontinuity.

Conservation law in one dimension:

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0$$

T = conserved density
X = flux of T

For us, $T = u$, $X = \frac{1}{2}u^2$.

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0.$$

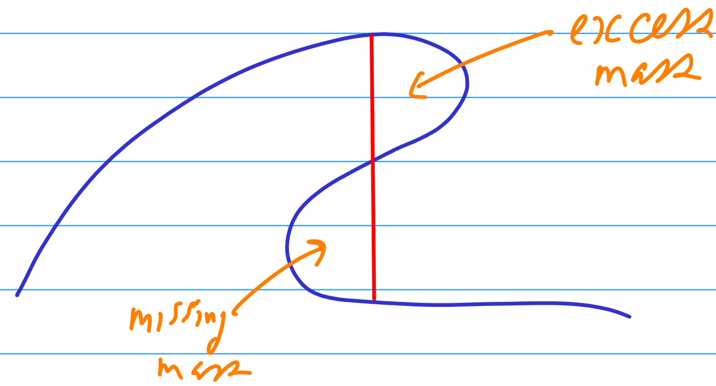
$$\frac{d}{dt}\int_a^b T(x)\,dx = -X\Big|_{x=a}^{b}$$

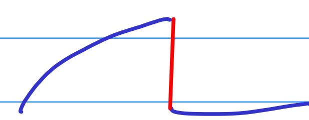
integrated form

A shock is a discontinuity in $u(t,x)$.

We assume that mass remains conserved even after the shock. (Pretty reasonable!)

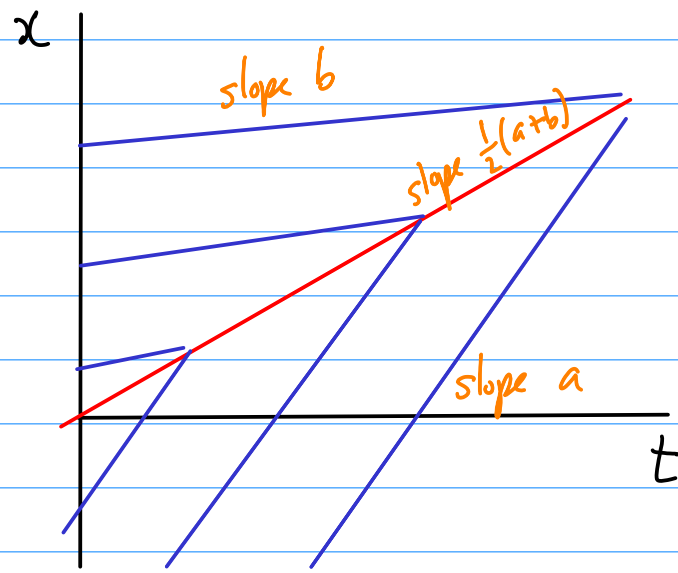
Equal area rule:





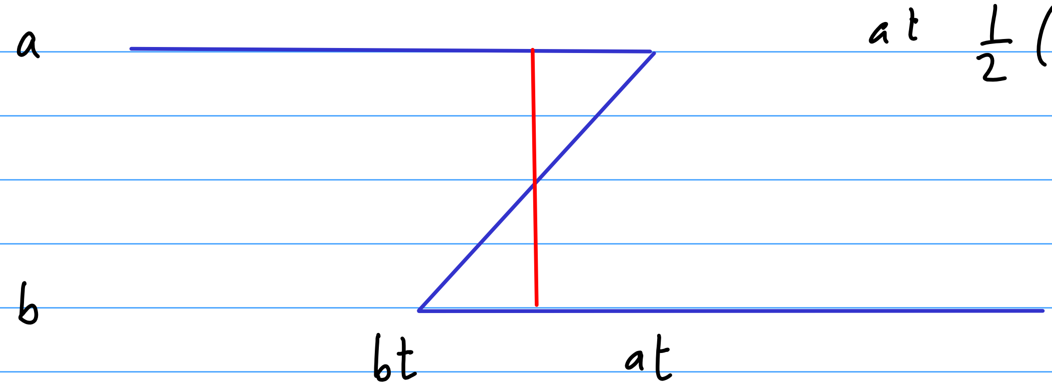
<u>example:</u> step wave: $u(0,x) = \begin{cases} a & x < 0 \\ b & x > 0 \end{cases}$

$a > b \Rightarrow$ already a shock wave.



Multiply-valued in

$bt < x < at$



Equal area:
draw shock
at $\frac{1}{2}(a+b)t$.

Shock-wave solution: $u(t,x) = \begin{cases} a, & x < ct \\ b & x > ct \end{cases}$

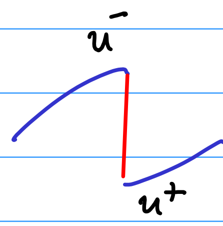
weak solution

$c = \frac{1}{2}(a+b)$

Shock speed is the average of solution values on either side:

Rankine-Hugoniot condition

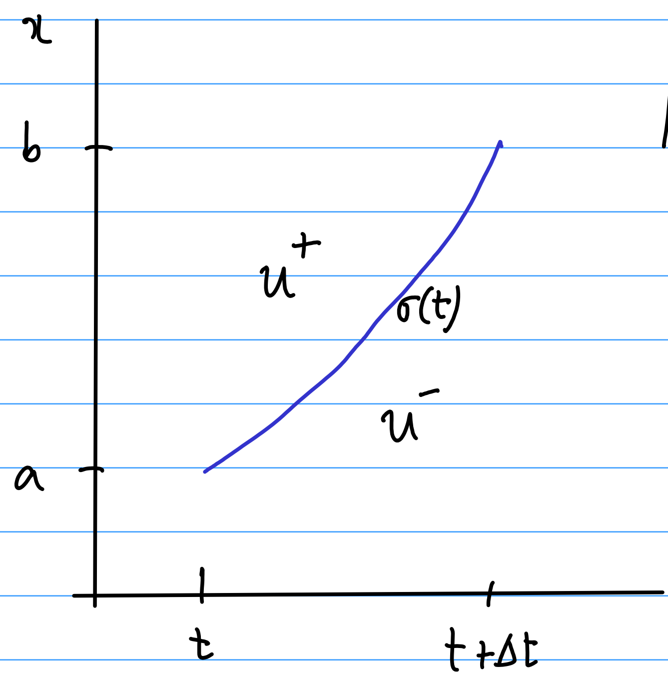
Prop: $u(t,x)$ solution to NL transport eq'n.



Discontinuity at $x = \sigma(t)$.

Then conservation of mass dictates $\dfrac{d\sigma}{dt} = \dfrac{1}{2}\left(u^+(t) + u^-(t)\right)$.

proof:



Mass in $[a,b]$ before shock.

$$M(t) = \int_a^b u(t,x)\,dx$$

$$\approx u^+(t)(b-a)$$

$$= u^+(t)\left(\sigma(t+\Delta t) - \sigma(t)\right)$$

$$= u^+(t)\,\sigma'(t)\,\Delta t$$

After: $M(t+\Delta t) = \displaystyle\int_a^b u(t+\Delta t, x)\,dx \approx u^-(t+\Delta t)\,(b-a)$

$$= u^-(t+\Delta t)\left(\sigma(t+\Delta t) - \sigma(t)\right)$$

$$= u^-(t)\,\sigma'(t)\,\Delta t$$

$$\frac{dM}{dt} = \lim_{\Delta t \to 0} \frac{M(t+\Delta t) - M(t)}{\Delta t} = [u^-(t) - u^+(t)]\,\sigma'(t)$$

On the other hand, the mass flux into $[a, b]$ is

$$\frac{dM}{dt} = \frac{1}{2}u^2(\tau, a) - \frac{1}{2}u^2(\tau, b), \quad \tau \in (t, t+\Delta t)$$

$$\to \frac{1}{2}u^{+2}(t, a) - \frac{1}{2}u^{-2}(t, b), \quad \tau \to t \text{ as } \Delta t \to 0$$

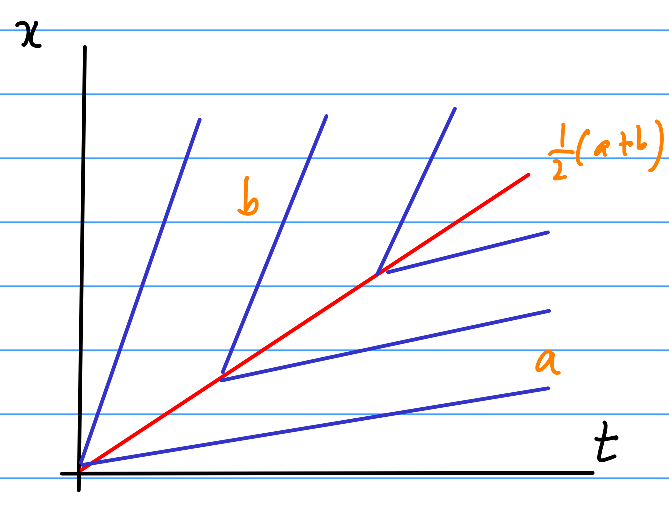
Hence, $\displaystyle\frac{dM}{dt} = \frac{1}{2}u^{+2}(t, a) - \frac{1}{2}u^{-2}(t, b)$

$$= [u^-(t) - u^+(t)]\,\sigma'(t)$$

which gives the result. ▨

<u>example:</u> rarefaction shock

$$u(0, x) = \begin{cases} a, & x > 0 \\ b, & x < 0 \end{cases} \qquad \underline{\underline{a < b}}$$

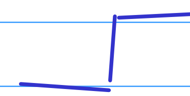


Shock "emits" characteristics!

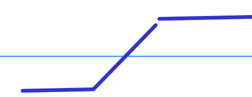
No data for $u(t,x)$ for $at < x < bt$.

A standard method is to fill in the gap with a "similarity solution":

$$u(t,x) = \frac{x}{t}$$



$$u(t,x) = \begin{cases} a, & x \le at \\ x/t, & at \le x \le bt \\ b, & x \ge bt \end{cases}$$



Is this right? In this case, since there is no real shock we can check by making sure a smoothed out initial condition converges to this. (2.3.13)

Causality thus requires that a shock satisfy

$$u^-(t) > \sigma'(t) = \frac{1}{2}(u^+ + u^-) > u^+(t)$$

Entropy condition: energy $\left(\frac{1}{2}u^2\right)$ only flows from "hot" to "cold"

$$\frac{dM}{dt} = \frac{1}{2}\left[u^{-2} - u^{+2}\right] > 0$$

Also: $u_t + c(u) u_x = 0$

Note: possible to define shocks that conserve other quantities (such as energy) rather than mass (momentum).