Lecture 4: Nonlinear transport

$$u_t + u u_x = 0. \quad \text{NONLINEAR!}$$

$c = u$: larger waves overtake smaller ones!

Try to solve as before: $\dfrac{dx}{dt} = u(t,x)$

In principle we can do this, though seems dodgy since we don't know $u(t,x)$.

Assume $u(t,x)$ solves the PDE, and let $h(t) = u(t, x(t))$.

As before: $\dfrac{dh}{dt} = \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} = 0$!

So h is constant along characteristics.

BUT: this means $\dfrac{dx}{dt} = u(t, x(t)) = \text{const}$!

So $x = ut + k$

Characteristic variable: $\xi = x - tu$.

So finally:

$$u = f(x - tu) \quad \text{implicit eq'n for } u$$

solves the PDE, with $f(\xi)$ a C^1 function.

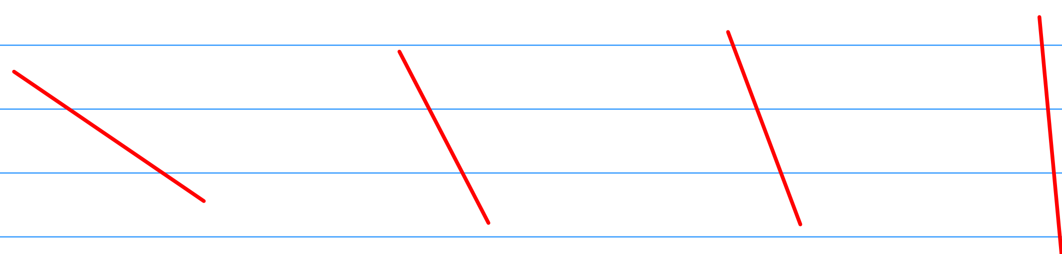
example: $f(\xi) = \alpha\xi + \beta$

straight line

$$u = \alpha(x - tu) + \beta \implies u(t,x) = \frac{\alpha x + \beta}{1 + \alpha t}$$

Blowup for $\alpha < 0$ at $t_* = -1/\alpha$.

$\alpha < 0$:



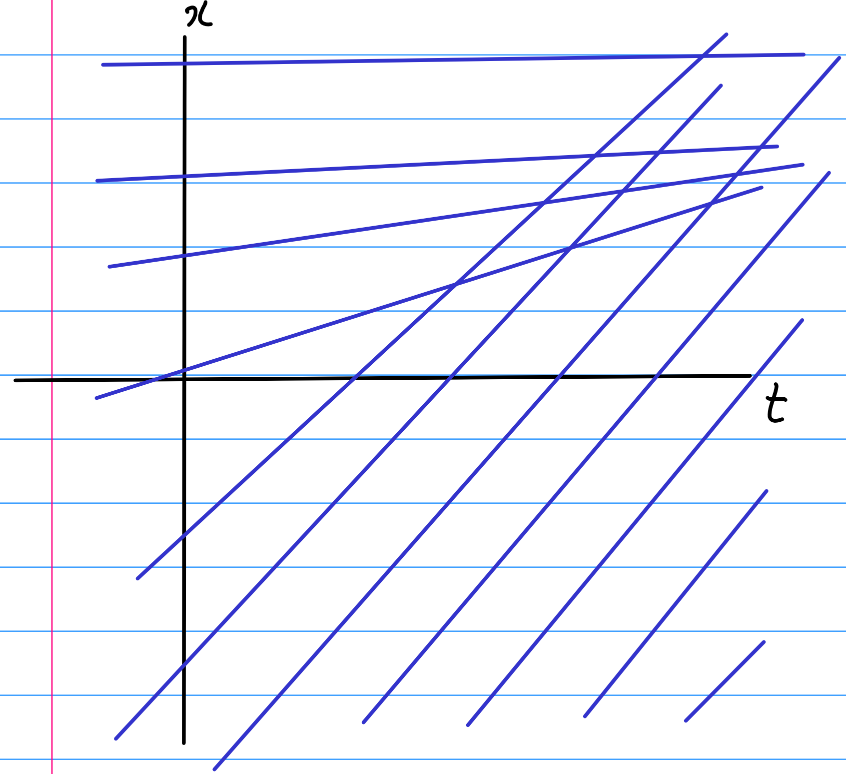
$t \to t_*$

We cannot legitimately continue beyond t_*.

For more general initial data: $u(0,x) = f(x)$

Draw char. line $x = t f(y) + y$ through $(0,y)$.

Then: $u(t, t f(y) + y) = f(y)$, for all t.



$$u(0,x) = \frac{\pi}{2} - \tan^{-1} x$$

$$u(0,-\infty) = \pi$$

$$u(0,\infty) = 0$$

Characteristics cross!

How to assign value?

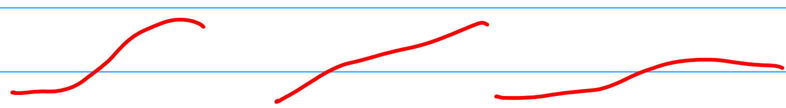
This kind of crossing does not happen if

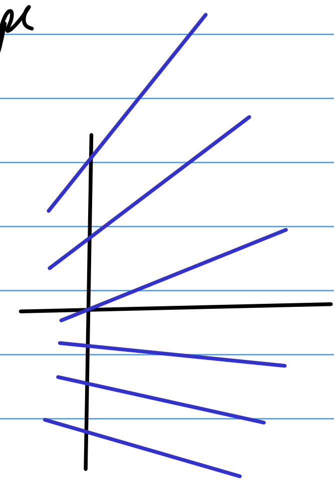
- All characteristics have same slope

or

- They "fan out": $f'(x) \geq 0$

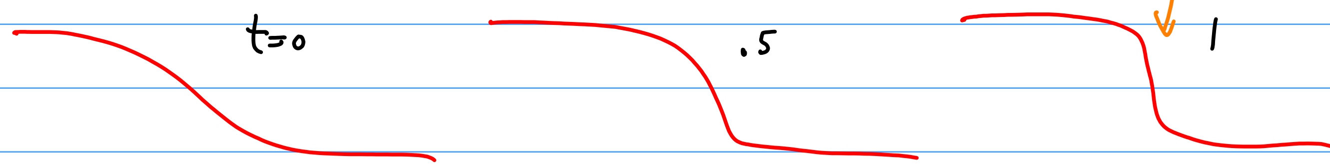
rarefaction wave

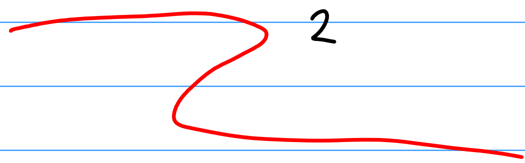




Go back to $u(0,x) = \frac{\pi}{2} - \tan^{-1} x$.

$\frac{\partial u}{\partial x} \to \infty$





multi-valued solution after t_*.

compression wave

When? $$u = f(x - tu) = f(\xi)$$

$$\frac{\partial u}{\partial x} = f'(\xi)\frac{\partial \xi}{\partial x} = f'(\xi)\left(1 - t\frac{\partial u}{\partial x}\right)$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{f'(\xi)}{1 + t f'(\xi)}}$$

Hence, $\frac{\partial u}{\partial x} \to \infty$ as $t \to -\frac{1}{f'(\xi)}$ (Need $f'(x) < 0$)

Earliest time this happens is:

$$t_* = \min\left\{ -\frac{1}{f'(x)} \;\middle|\; f'(x) < 0 \right\}$$

If min occurs for $x = x_0$, propagate forward to

$$x_* = x_0 + f(x_0)\, t_*$$

In our example, $f(x) = \frac{\pi}{2} - \tan^{-1} x$

$$f'(x) = -\frac{1}{1+x^2}, \quad \frac{-1}{f'} = 1+x^2$$

$$t_* = \min\{1+x^2\} = 1 \quad \text{with } x_0 = 0$$

$$x_* = 0 + f(0)\cdot 1 = \frac{\pi}{2}$$