Lecture 3: Transport equation (examples)

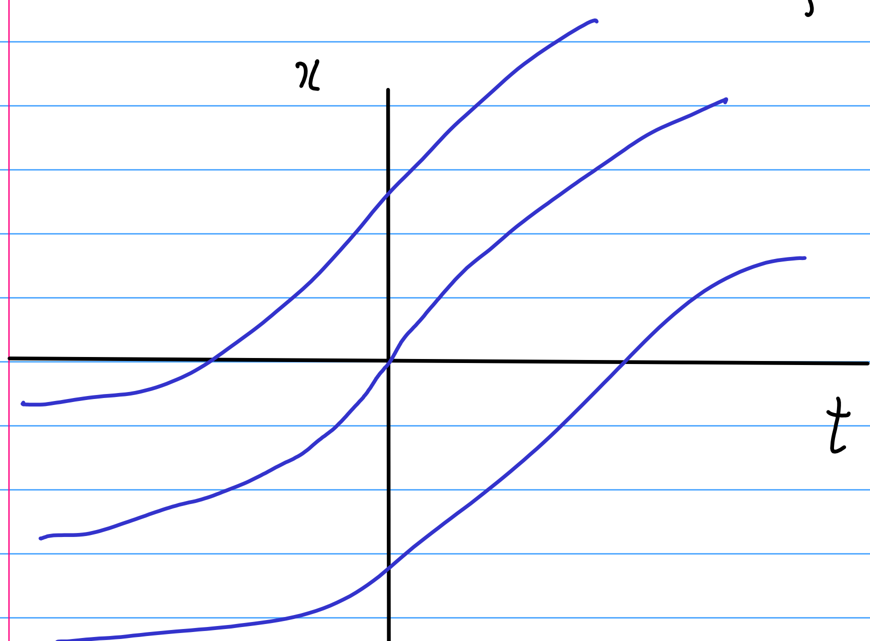
$$\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = 0 \qquad\qquad \frac{dx}{dt} = c(x)$$

$$\beta(x) = \int \frac{dx}{c(x)} = t + k$$

$$u(t,x) = f \circ \beta^{-1}\left(\beta(x) - t\right)$$

example:
$$\frac{\partial u}{\partial t} + \frac{1}{x^2+1}\frac{\partial u}{\partial x} = 0$$

$$\frac{dx}{dt} = \frac{1}{x^2+1} \implies \beta(x) = \int (x^2+1)\,dx$$
$$= \frac{1}{3}x^3 + x = t + k$$

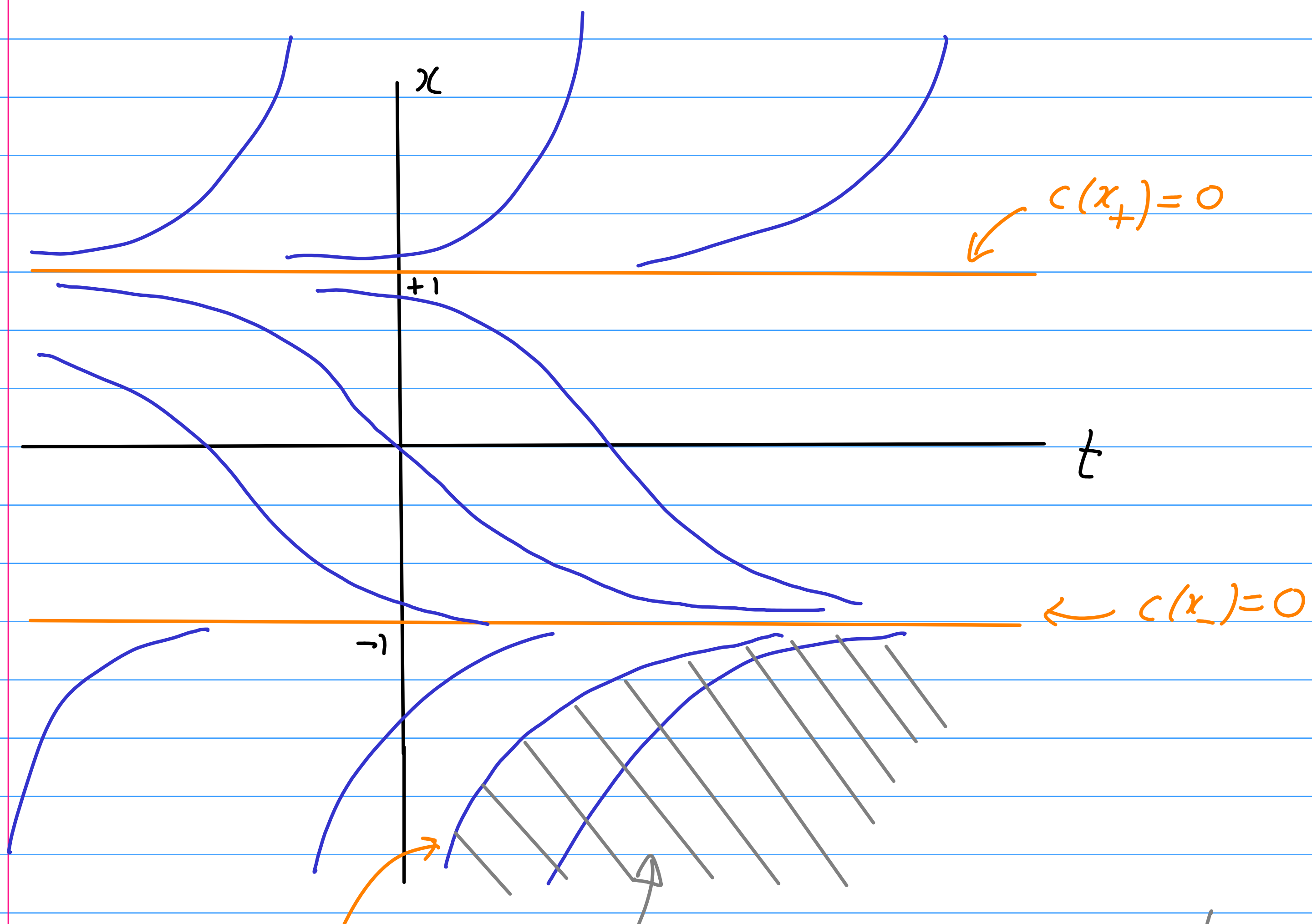


Parts of the wave speed up then slow down after they pass $x = 0$.

<u>example:</u> $u_t + (x^2 - 1)\, u_x = 0$

Characteristics: $\dfrac{dx}{dt} = x^2 - 1 = c(x)$

$$\beta(x) = \int \frac{dx}{x^2 - 1} = \frac{1}{2} \log \left| \frac{x-1}{x+1} \right| = t + k$$



$x(t) = \dfrac{1+e^{2t}}{1-e^{2t}}$, $t>0$

NOT prescribed by initial data!

Choose: set to 0

$x < -1$: (actually $|x| > 1$)

$$\beta(x) = \frac{1}{2}\log\left(\frac{x-1}{x+1}\right) = t + k$$

$$\frac{1-x}{1+x} = e^{2(t+k)}$$

$$\beta^{-1}(t+k) = x(t) = \frac{1 + e^{2(t+k)}}{1 - e^{2(t+k)}}, \quad x < -1$$

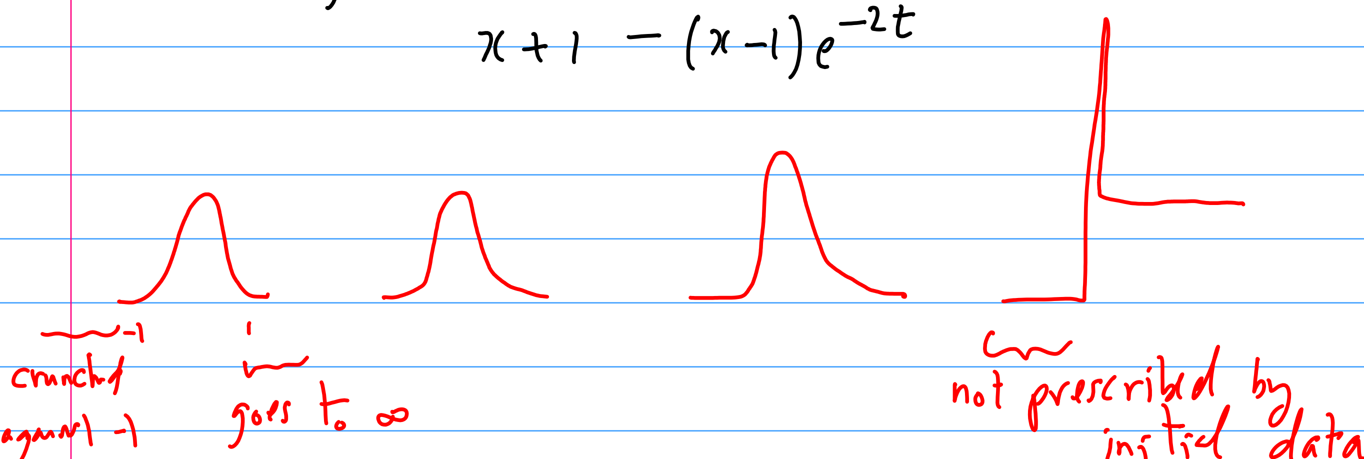
The char. that diverges to $-\infty$ at $t \to 0^+$ has $k = 0$.

With $u(0,x) = e^{-x^2} = f(x)$

$$u(t,x) = f \circ \beta^{-1}(\beta(x) - t)$$

$$= f \circ \frac{x-1+(x+1)e^{2t}}{1-x+(x+1)e^{2t}}$$ actually valid for all x

$$= f \circ \frac{x+1+(x-1)e^{-2t}}{x+1-(x-1)e^{-2t}}$$



Converges non-uniformly to a step function:

$$u(t,x) \xrightarrow[t\to\infty]{} s(x) = \begin{cases} f(1), & x \geq -1 \\ 0, & x < -1 \end{cases}$$

For $c(x)$ continuously differentiable:

- Unique char. through each $(t,x) \in \mathbb{R}^2$
- Cannot cross
- $t = \beta(x)$ char. $\Rightarrow$ $t = \beta(x) + h$ also a char.
- Each non-horizontal char. is graph of strictly monotone function. Never reverses direction
- As t increases, either $x(t) \xrightarrow{(t\to\infty)} x_*$ with $c(x_*) = 0$ or $x(t) \to \pm\infty$.