Lecture 2: Transport equation

$x \in D$

Stationary waves: $\frac{\partial u}{\partial t} = 0$, $u(t,x)$

Trivial! $u(t,x) = u(0,x)$. $\to C^1$

Except: maybe domain is disconnected

Transport equation: $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$

(Also: unidirectional wave equation.)

Let $\xi = x - ct$ "characteristic variable"

$$u(t,x) = v(t, x-ct) = v(t,\xi)$$

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial t} = \frac{\partial v}{\partial t} - c\frac{\partial v}{\partial \xi}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial v}{\partial \xi}$$

Holding ξ constant!

Hence: $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \frac{\partial v}{\partial t} = 0$

$v = v(\xi)$

We win: $u(t,x) = v(x-ct)$.

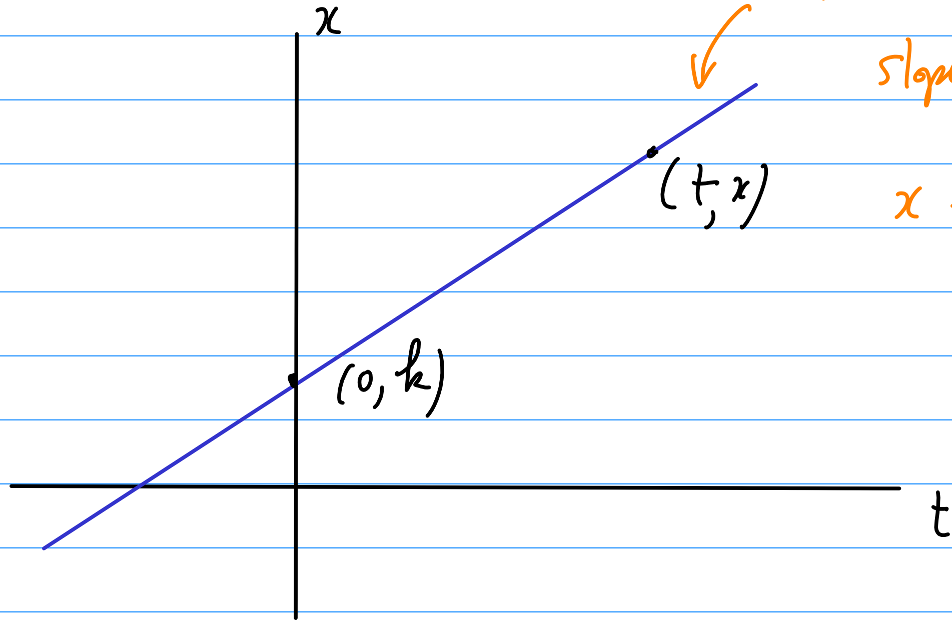
$v(\xi)$ is a C^1 function of ξ.

$u(0,x) = v(x) \Leftarrow$ initial profile.

The initial profile "travels" to the right with speed $c>0$.

"characteristic line"

slope $= c$



$x = ct + k$

Transport with decay: $\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = -au$

Again using ξ: $\dfrac{\partial v}{\partial t} = -av$ Again get ODE

Integrating factor:

$$e^{at}\left(\frac{\partial v}{\partial t} + av\right) = \frac{\partial}{\partial t}\left(e^{at} v\right) = 0$$

$$v(t,\xi) = f(\xi)\, e^{-at}$$

$$u(t,x) = f(x - ct)\, e^{-at}$$

moving decaying

We can say that the value of u decays along characteristics.

<u>Nonuniform transport:</u> $\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = 0$

Characteristics are no longer straight lines.

Write $x(t)$ for characteristic.

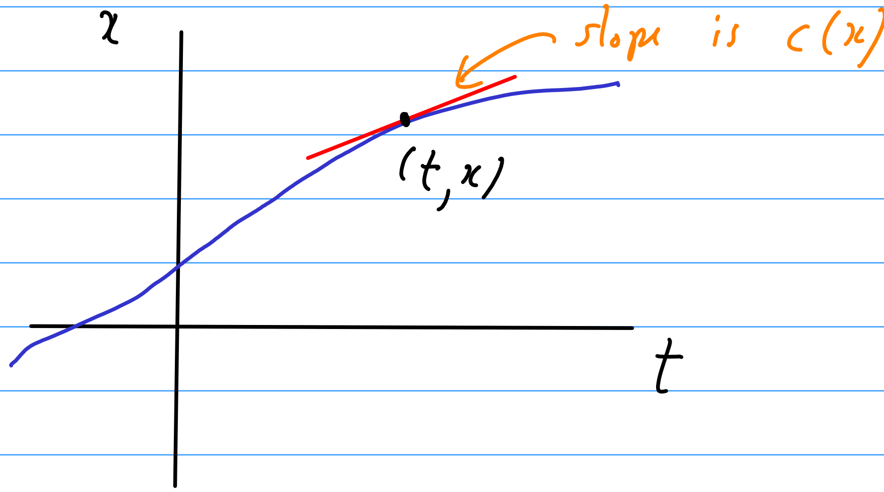
$h(t) = u(t, x(t))$ value of u along characteristic

$$\frac{dh}{dt} = \frac{\partial u}{\partial t}(t, x(t)) + \frac{\partial u}{\partial x}(t, x(t))\,\frac{dx}{dt}$$

Choose: $\frac{dx}{dt} = c(x)$ ODE

Then: $\frac{dh}{dt} = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$!!

$x(t)$ is a <u>characteristic curve</u> for the transport eq!



We can formally solve $\frac{dx}{dt} = c(x)$:

$$\frac{dx}{c(x)} = dt \iff \beta(x) = \int \frac{dx}{c(x)} = t + k$$

← $c(x) \neq 0$!

$$x(t) = \beta^{-1}(t + h)$$ (inverse function)

$u(t, x)$ is constant along char. curves;

Let $\xi = \beta(x) - t$. ($\xi = \frac{x}{c} - t$ for constant $c(x)$.)

Then $u(t, x) = v(\xi) = v(\beta(x) - t)$

Thus, to solve with initial value $u(0,x) = f(x)$,

$$v(\beta(x)) = f(x) \Rightarrow v(\xi) = f \circ \beta^{-1}(\xi)$$

$$u(t,x) = f \circ \beta^{-1}(\beta(x) - t)$$