Lecture 26: WKB Theory

$$\varepsilon y'' + y = 0, \quad y(0)=0, \; y(1)=1$$

$$y(x) = \frac{\sin(x/\sqrt{\varepsilon})}{\sin(1/\sqrt{\varepsilon})} \qquad (\sqrt{\varepsilon} \neq (n\pi)^{-1})$$

Global breakdown as $\varepsilon \to 0^+$

Boundary layer analysis doesn't work.

Try: $\quad y(x) \sim A(x)\, e^{S(x)/\delta}, \quad \delta \to 0^+$ as $\varepsilon \to 0^+$

$S(x)$ is the (possibly imaginary) phase. Varies slowly in breakdown region.

$S(x)$ real $\to$ boundary layer
imag. $\to$ fast oscillation

Formal WKB expansion:

$$y(x) \sim \exp\left[\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n(x)\right]$$

example: $\quad \varepsilon^2 y'' = Q(x)\, y \qquad Q(x) \neq 0$

$$y' \sim \frac{1}{\delta} \sum_{n=0}^{\infty} \delta^n S_n' \; y$$

$$y'' \sim \left[\frac{1}{\delta^2} \left(\sum_{n=0}^{\infty} \delta^n S_n' \right)^2 + \frac{1}{\delta} \sum_{n=0}^{\infty} \delta^n S_n'' \right] y$$

$$\Rightarrow \underbrace{\frac{\varepsilon^2}{\delta^2} (S_0')^2}_{} + \frac{2\varepsilon^2}{\delta} S_0' S_1' + \frac{\varepsilon^2}{\delta} S_0'' + \cdots = Q(x)$$

largest term → must balance $Q(x)$ [dominant balance]

Hence, $\delta = \varepsilon$, and $(S_0')^2 = Q(x)$ eikonal eq'n

$$\Rightarrow S_0(x) = \pm \int^x \sqrt{Q(t)}\, dt$$ up to additive constants

Next order: $2 S_0' S_1' + S_0'' = 0$ transport eq'n

$$\Rightarrow S_1(x) = -\frac{1}{4} \log Q(x)$$

General solution:

$$y(x) \sim c_1 Q^{-1/4}(x) \exp\left[\frac{1}{\varepsilon} \int_a^x \sqrt{Q(t)} \right]$$

$$+ c_2 Q^{-1/4}(x) \exp\left[-\frac{1}{\varepsilon} \int_a^x \sqrt{Q(t)} \right]$$

$$+ \cdots, \quad \varepsilon \to 0$$

This is the leading-order WKB solution.

For a WKB series to be useful, we require that it be an asymptotic series as $\delta \to 0$ for all x:

$$\delta^n S_{n+1}(x) \ll \delta^{n-1} S_n(x), \quad \delta \to 0$$

uniformly in x

Optimal truncation rule: truncate series before the smallest term $\delta^N S_{N+1}(x) \ll 1$, all x as $\delta \to 0$.

[Because then $\exp[\delta^N S_{N+1}(x)] \sim 1 + O(\delta^N S_{N+1}(x))$,

so $\dfrac{y(x) - \exp\left(\frac{1}{\delta}\sum \delta^n S_n\right)}{y(x)} \sim O(\delta^N S_{N+1}(x))$]

Levels of approximation:

$$y(x) \approx e^{S_0(x)/\delta}$$ geometrical optics (not asymptotic)

$$y(x) \sim e^{S_0(x)/\delta + S_1(x)}$$ physical optics

example: Airy equation $y'' = xy$

Insert ε: $\varepsilon y'' = xy$

large x is like small ε

WKB solution: $S_0 = \pm \frac{2}{3} x^{3/2}$

$$S_1 = -\frac{1}{4} \log x$$

$$S_2 = \pm \frac{5}{48} x^{-3/2}$$

ε^0 $\quad y \simeq \exp\left(\pm \frac{2}{3} x^{3/2}\right)$ $\quad (\varepsilon = 1)$ $\quad x \to \infty$

ε^1 $\quad y \sim x^{-1/4} \exp\left(\pm \frac{2}{3} x^{3/2}\right)$ $\leftarrow$ asymptotic

small

ε^2 $\quad y \sim x^{-1/4} \exp\left(\pm \frac{2}{3} x^{3/2}\right) \exp\left(\pm \frac{5}{48} x^{-3/2}\right)$

$$\sim x^{-1/4} \exp\left(\pm \frac{2}{3} x^{3/2}\right)\left(1 \pm \frac{5}{48} x^{-3/2}\right)$$

also asymptotic

The order of the equation matters little:

$$\varepsilon^n \frac{d^n}{dx^n} y = Q(x) y$$

$$\Rightarrow S_0 = \omega \int^x [Q(t)]^{1/n} dt, \quad \omega^n = 1.$$

$$S_1 = \frac{1-n}{2n} \log Q$$

Note that if $Q(x) = 0$ somewhere, $\frac{S_1}{S_0/\varepsilon} \cancel{\ll} 1$ and no longer asymptotic.

Points where $Q(x)$ vanishes are called turning points

requires matched asymptotics

Another example:

$$\varepsilon y'' + a(x) y' + b(x) y = 0, \quad y(0) = A, \; y(1) = B$$

(we've done this before!)

$a(x) > 0, \; 0 \leq x \leq 1$

$$\frac{\varepsilon}{\delta^2} S_0'^2 + 2\frac{\varepsilon}{\delta} S_0' S_1' + \frac{\varepsilon}{\delta} S_0'' + \frac{1}{\delta} S_0' a + S_1' a + \dots = 0$$

dominant balance $\quad \delta = \varepsilon$

$$S_0'^2 + S_0' a = 0, \qquad 2 S_0' S_1' + S_0'' + S_1' a + b = 0$$

S_0 either: $S_0' = 0$ or $S_0' = -a$

$$S_1' a + b = 0 \qquad\qquad a S_1' + a' = b$$

$$S_1 = -\int^x \frac{b(t)}{a(t)}\, dt \qquad\qquad S_1 = -\log a + \int_0^x \frac{b(t)}{a(t)}\, dt$$

These give, respectively:

$$y_1(x) \sim c_1 \exp\left[-\int_0^x \frac{b(t)}{a(t)}\,dt\right]$$

outer solution!

or

$$y_2(x) \sim c_2 \frac{1}{a(x)} \exp\left[\int_0^x \frac{b(t)}{a(t)}\,dt - \frac{1}{\varepsilon}\int_0^x a(t)\,dt\right]$$

inner solution!

Then we fix the constants using boundary conditions:

$$A = c_1 + \frac{c_2}{a(0)}, \qquad B = c_1 \exp\left[-\int_0^1 \frac{b(t)}{a(t)}\,dt\right]$$

where we neglected $\exp\left[-\frac{1}{\varepsilon}\int_0^1 a(t)\,dt\right]$ exponentially small

Thus:

$$y(x) \sim B \exp\left[\int_x^1 \frac{b}{a}\,dt\right] + \frac{a(0)}{a(x)}\left[A - B\exp\int_0^1 \frac{b}{a}\,dt\right] \times \exp\left[\int_0^x \frac{b}{a}\,dt - \frac{1}{\varepsilon}\int_0^x a\,dt\right]$$

makes 2nd term negligible unless x is small, but then neglect

Hence, as $\varepsilon \to 0$:

$$y(x) \sim B\exp\left[\int_x^1 \frac{b}{a}\,dt\right] + \frac{a(0)}{a(x)}\left[A - B\exp\int_0^1 \frac{b}{a}\,dt\right] e^{-a(0)x/\varepsilon}$$

to leading order. This is our y_{unif} from before.