Lecture 25: Boundary layer theory (cont'd)

B&O 9.4

So far our boundary layers have been of thickness

$$\delta \sim \varepsilon.$$

Let's be more explicit about determining this:

Recall the singular problem:

$$\varepsilon y'' + a(x)y' + b(x)y = 0, \quad y(0) = A, \; y(1) = B$$

with $a(x) > 0$, $0 \leq x \leq 1$. This has a B. layer at $x = 0$.

Inside the layer, let $y(x) = Y_{in}(X)$, $X = x/\delta$:

$$\frac{\varepsilon}{\delta^2}\frac{d^2 Y_{in}}{dX^2} + \frac{a(\delta X)}{\delta}\frac{dY_{in}}{dX} + b(\delta X)Y_{in} = 0$$

We wish to determine $\delta(\varepsilon)$. There three possibilities:

① $\delta(\varepsilon) \ll \varepsilon$, ② $\delta(\varepsilon) \sim \varepsilon$, ③ $\delta(\varepsilon) \gg \varepsilon$ as $\varepsilon \to 0$.

For ①, get $\dfrac{d^2 Y_{in}}{dX^2} = 0$, since $\dfrac{\varepsilon}{\delta^2} \gg 1$ and

$$\frac{a/\delta}{\varepsilon/\delta^2} \sim \frac{a\delta}{\varepsilon} \ll 1$$

Thus in that case $Y_{in} = A + cX$. This cannot match $y_{out}(0)$ as $X \to \infty$, since $y_{out}(0)$ is finite, unless $c = 0$. But then need $y_{out}(0) = A$, which is not true in general. Thus we rule out 1.

For case 3, $1 \gg \delta(\varepsilon) \gg \varepsilon$, we get the equation

$$a(0) \frac{dY_{in}}{dX} = 0 \quad , \quad \text{so} \quad Y_{in} = A \neq y_{out}(0)$$

This doesn't work either.

Finally try: 2 $\delta(\varepsilon) \sim \varepsilon$:

$$\frac{d^2 Y_{in}}{dX^2} + a(0) \frac{dY_{in}}{dX} = 0$$

This is a distinguished limit because it involves >1 term, balancing each other out.

As we saw earlier, this is the correct limit, which allows us to match.

Let's do another example:

$$\varepsilon y'' - x^2 y' - y = 0, \qquad y(0) = y(1) = 1.$$

Write the outer solution as $y_{out} = y_0 + \varepsilon y_1 + \dots$

Then: $\quad -x^2 y_0' - y_0 = 0$, so $y_0 = C_0 e^{1/x}$.

Clearly we can satisfy the B.C. at $x=1$.

$$y_0(1) = C_0 e = 1 \Rightarrow C_0 = e^{-1}.$$

Hence, $y_0(x) = e^{\frac{1}{x}-1}$.

But this cannot satisfy $y(0)=0$, since $y_0(0) \to \infty$.

Need boundary layer near $x=0$.

Let $X = x/\delta$, $y(x) = Y(X)$:

$$\frac{\varepsilon}{\delta^2} Y'' - \frac{\delta^{\cancel{2}} X^2}{\cancel{\delta}} Y' - Y = 0$$

Possible balance: $1 \ll \varepsilon \ll \delta$: $\Rightarrow Y = 0$ (no match)

$\varepsilon \sim \delta$: $\Rightarrow Y'' = 0$ (no match)

$1 \ll \delta \ll \varepsilon$: $\Rightarrow$ distinguished limit with $\delta \sim \varepsilon^{1/2}$.

With $\delta \sim \varepsilon^{1/2}$,

$$Y'' - Y = 0 \Rightarrow Y \sim e^{\pm X}$$

But there is a matching issue here as well: the outer solution $\sim e^{1/x}$ as $x \to 0$, whereas at best $Y \sim e^{X}$. There is no way to match these.

$\longrightarrow$ We missed something!

What we missed is a boundary layer at $x=1$.

Go back to outer: $y_0 = C_0 e^{1/x}$.

careful! we re-use variables

Check near $x=1$; let $X = \dfrac{1-x}{\delta}$, $y(x) = Y(X)$:

$y' = -\frac{1}{\delta}Y'$

$$\frac{\varepsilon}{\delta^2} Y'' + \frac{(1-\delta X)^2}{\delta} Y' + Y = 0.$$

$y'' = \frac{1}{\delta^2}Y''$

To leading order, $\dfrac{(1-\delta X)^2}{\delta} \sim \dfrac{1}{\delta}$.

There is a distinguished limit for $\delta \sim \varepsilon$, for then

$$Y_{in}'' + Y_{in}' = 0$$

right ($x=1$)

Hence, $Y_{in} = A_0 + B_0 e^{-X}$

with $y(1) = Y_{in}(0) = A_0 + B_0 = 1$.

Matching to $y_0(1)$ requires $\lim_{X\to\infty} Y_{in}(X) = A_0 = C_0 e$.

So $Y_{in}(X) = eC_0 + (1-eC_0)e^{-X}$.

However, we are not done! We need to go back to the boundary layer near $x=0$.

For the left boundary layer we found the distinguished limit

$$\delta \sim \varepsilon^{1/2}, \quad Y_{in}'' - Y_{in} = 0 \qquad \text{left } (x=0)$$

$$Y_{in} = D_0 e^{X} + E_0 e^{-X} \qquad \text{Here } X = x/\varepsilon^{1/2}$$

$$Y_{in}(0) = 1 \implies D_0 + E_0 = 1$$

Match to outer: as $X \to \infty$, $Y_{in} \sim D_0 e^{X}$

as $x \to 0$, $y_0 \sim C_0 e^{1/x}$

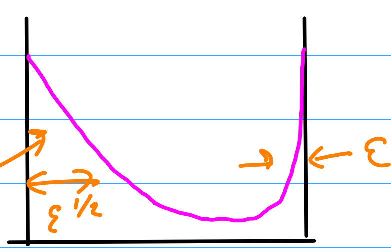
These cannot be matched, unless $C_0 = D_0 = 0$!

$$\text{So } \begin{cases} Y_{in} = e^{-X} = e^{-x/\varepsilon^{1/2}} & \text{(on the left)} \\ Y_{in} = e^{-X} = e^{-(1-x)/\varepsilon} & \text{(on the right)} \\ y_0 = 0 \end{cases}$$

The uniform approximation is then

$$y_{unif} = \overset{0}{y_0} + \left(Y_{in,\,left} - \overset{0}{y_{left\ match}}\right) + \left(Y_{in,\,right} - \overset{0}{y_{right\ match}}\right)$$

$$= e^{-x/\varepsilon^{1/2}} + e^{(x-1)/\varepsilon}$$



See B&O p. 439-440 for plots

B & O p. 442

Logarithmic corrections can also arise:

Consider: $\varepsilon y'' + xy' - xy = 0, \quad y(0) = 0, \; y(1) = e.$

As in our previous example, there is a B. layer of thickness $\varepsilon^{1/2}$ at $x=0$:

$$\frac{\varepsilon}{\delta^2} Y'' + XY' - \delta XY = 0$$

distinguished limit is $\delta^2 \sim \varepsilon$

For $\delta = \varepsilon^{1/2}$, $\quad Y_{in}'' + XY_{in}' = \varepsilon^{1/2} XY_{in}$

look for series solution in powers of $\varepsilon^{1/2}$? No, see below

Outer:

$$\text{Let } y_{out}(x) = \sum_{n=0}^{\infty} y_n(x)\, \varepsilon^n$$

Satisfies: $y_0' - y_0 = 0, \quad y_n' - y_n = -\frac{1}{x} y_{n-1}'' \quad (n>0)$

This has solution

$$y_{out}(x) = e^x - \varepsilon e^x \log x + \varepsilon^2 e^x \left[\frac{1}{2}(\log x)^2 - \frac{2}{x} + \frac{1}{2x^2} + \frac{3}{2} \right] + \ldots$$

The log terms (singular as $x \to 0$) will require the inner expansion to also contain log corrections.

[See B & O, p. 446]