Lecture 24: Boundary layer theory (cont'd)

A quick note about convergence:

exponential integral

Consider
$$I(x) = x e^{x} \int_x^{\infty} \frac{e^{-t}}{t}\, dt \quad , \quad x \geq 0$$

Let's try to approximate $I(x)$, for large x.

See Bleistein & H Sec 1.1

Repeated integration by parts gives
$$I(x) = \sum_{n=0}^{N-1} \frac{(-1)^n n!}{x^n} + (-1)^N N!\, x e^{x} \int_x^{\infty} \frac{e^{-t}}{t^{N+1}}\, dt$$
$$= S_N(x) + \varepsilon(x,N)$$

Now note that ratio test gives $\left| \frac{(n+1)\text{th term}}{n\text{th term}} \right| = \frac{n}{x}$
for $S_N(x)$, so for fixed x diverges for all x!

So throw out $S_N(x)$? No!

Observe that $\varepsilon(x,N) > 0 \;,\; N$ even
$\qquad\qquad\qquad\quad < 0 \;,\; N$ odd

$$I(x) = S_N(x) + |\varepsilon(x,N)| \quad , \quad N \text{ even}$$

$$= S_{N+1}(x) - |\varepsilon(x,N+1)|$$

So

$$S_N(x) \leq I(x) \leq S_{N+1}(x) \quad , \quad N \text{ even}$$

Hold on here: this says that I can approximate $I(x)$ by partial sums of S_N, even though S_N diverges as $N \to \infty$!

The optimal approximation is the one that minimizes $\varepsilon(x,N)$. This error gets smaller as x gets larger. This gives meaning to

$$I(x) \simeq 1 - \frac{1}{x}$$

It means the approximation holds for x small enough, but adding more terms doesn't necessarily improve the approximation (unless x is much smaller)

Thus, typically divergent series are useful for approximations!

example: $\quad I(100) \simeq 0.99019$

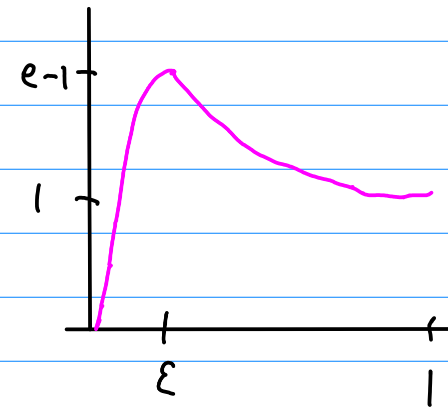
$$1 - \frac{1}{100} = 0.99$$

Back to boundary layers

Let's formalize the concepts better.

Consider: $\varepsilon y'' + (1+\varepsilon) y' + y = 0$, $\quad y(0)=0$, $y(1)=1$

Exact solution: $$y = \frac{e^{-x} - e^{-x/\varepsilon}}{e^{-1} - e^{-1/\varepsilon}}$$



There is a boundary layer of width $\delta = \varepsilon$.

Outer limit: $y_{out}(x) = \lim_{\varepsilon \to 0^+} y(x) = e^{1-x}$

This works at fixed x.

same solution

Directly in the equation: $y_{out}' + y_{out} = 0$, $\quad y_{out}(1) = 1$

For the inner solution take $\varepsilon \to 0$, but for x values always inside the boundary layer.

$$y_{in}(x) = Y_{in}(X) = \lim_{\varepsilon \to 0^+} y(\varepsilon X) = e - e^{1-X}$$

thickness of layer

where $x = \varepsilon X$

inner variable

Directly from equation, rewrite in terms of $Y(X) = y(\varepsilon x)$:

$$\frac{1}{\varepsilon^2}\frac{d^2 Y}{dX^2} + \left(\frac{1}{\varepsilon} + 1\right)\frac{dY}{dX} + Y = 0$$

Take $\varepsilon \to 0^+$, with X fixed:

$$\frac{d^2 Y_{in}}{dX^2} + \frac{dY_{in}}{dX} = 0 \quad , \quad Y_{in}(0) = 0$$

$Y_{in} = e - e^{1-X}$ satisfies this.

match!

Note that $\lim\limits_{x\to 0} y_{out}(x) = \lim\limits_{X\to\infty} Y_{in}(X) = e$

In general the limit is not a number, but some function

Go to higher order:

$$y_{out}(x) \sim \sum_{n=0}^{\infty} y_n(x)\,\varepsilon^n \quad , \quad \varepsilon \to 0^+$$

formal asymptotic series

$y_0(1) = 1, \quad y_n(1) = 0, \quad n > 0.$

First find $y_{out}(x)$ perturbatively.

$$y_0' + y_0 = 0 \quad , \quad y_0(1) = 1$$

$$y_n' + y_n = -y_{n-1}'' - y_{n-1}' \quad , \quad y_n(1) = 0, \; n > 0$$

Solution is. $y_0 = e^{1-x}$, $y_n = 0$, $n > 0$.

So in this case the leading-order outer solution from before is correct to all orders in ε.

$$|y_{out} - y| \sim O(\varepsilon^n) \text{ for all } n$$

Now for the inner solution:

$$Y_{in}(X) \sim \sum_{n=0}^{\infty} \varepsilon^n Y_n(X), \quad \varepsilon \to 0^+$$

with $Y_n(0) = 0$, all n.

$$Y_0'' + Y_0' = 0$$

$$Y_n'' + Y_n' = -Y_{n-1}' - Y_{n-1}$$

$$Y_0(X) = A_0 (1 - e^{-X})$$

$$\Rightarrow \quad Y_n(X) = \int_0^X \left(A_n e^{-z} - Y_{n-1}(z) \right) dz, \quad n > 0$$

The A_n are undetermined constants.

Matching: substitute $x = \varepsilon X$ into y_{out}:

$$y_{out}(\varepsilon X) = e^{1-\varepsilon X} = e\left[1 - \varepsilon X + \frac{\varepsilon^2 X^2}{2!} - \frac{\varepsilon^3 X^3}{3!} + \dots\right]$$

$$Y_0(X) = A_0 \quad \text{as } X \to \infty$$

So $A_0 = e$, to match with $y_{out}(x)$.

$$Y_1(X) = (A_1 + A_0)(1 - e^{-X}) - eX$$

$$= A_1 + A_0 - eX \quad \text{as } X \to \infty$$

$= -eX$ from matching with y_{out} to order ε,

So $A_1 = -A_0 = -e$.

etc... Get eventually $Y_{in}(X) = e \sum_{n=0}^{\infty} \frac{\varepsilon^n (-1)^n X^n}{n!} - e^{1-X}$

$$= e^{1-\varepsilon X} - e^{1-X}$$

Uniformly valid solution: $\quad = e^{1-x} - e^{1-x/\varepsilon}$

$y_{unif} = y_{in} + y_{out} - y_{match}$

$$= (e^{1-x} - e^{1-x/\varepsilon}) + (e^{1-x}) - (e^{1-x})$$

$$= e\left[e^{-x} - e^{-x/\varepsilon}\right]$$

Not the same as exact solution!

to all orders in ε

$$\frac{y_{exact}}{y_{unif}} = \left(\frac{e^{-x} - e^{-x/\varepsilon}}{e^{-1} - e^{-1/\varepsilon}}\right)\frac{e^{-1}}{e^{-x} - e^{-x/\varepsilon}}$$

$$= \left(1 - e^{1-1/\varepsilon}\right)^{-1}$$

$$\simeq 1 + e^{1-1/\varepsilon}$$

Correction is "beyond all orders" in ε