Lecture 23: Boundary-layer theory

example: $(x - \varepsilon y)y' + xy = e^{-x}$, $y(1) = 1/e$, find $y(0)$ to leading order in ε

nonlinear

Set $\varepsilon = 0$: $x(y' + y) = e^{-x}$

$$y = (1 + \log x)\, e^{-x}$$

But as $x \to 0$, $y \to \infty$, so $\varepsilon y y' \to \infty$.

There is a boundary layer near $x = 0$.

Split in two regions: "inner" (inside boundary layer)
"outer" (the rest)

$$y_{out} = (1 + \log x)\, e^{-x}$$ since $y(1) = 1/e$ is in outer region.

Assume boundary layer of width δ.

$\varepsilon\, y_{out}\, y_{out}' \sim \varepsilon \ln \delta \left(\frac{1}{\delta}\right) \sim O(1)$, so $\dfrac{\delta}{\log \delta} \sim O(\varepsilon)$

Note that $\dfrac{\delta}{\log \delta} \to 0$ as $\varepsilon \to 0$

In the inner region $x \ll \delta$, use small x:

$$(x - \varepsilon y_{in}) y_{in}' = 1$$

We used: $\frac{xy}{xy'} \sim \frac{y}{y'} \ll 1$ since y blows as $\varepsilon \to 0$

Solution: $x = \varepsilon (y_{in} + 1) + C e^{y_{in}}$

Can use this to determine $y(0)$.

To find C, use asymptotic matching.

Take x small, but not as small as δ: $x = O(\varepsilon^{1/2})$ (say)

$$\left.\begin{aligned} y_{out} &\sim 1 + \ln x \\ x &\sim C e^{y_{in}} \end{aligned}\right\} \quad C = 1/e$$

transcendental

Thus: $y_{in}(0)$ satisfies $0 = \varepsilon (y_{in}(0) + 1) + e^{y_{in}(0) - 1}$

Clearly, $y_{in}(0) \to -\infty$ as $\varepsilon \to 0$.

Let $\alpha = -y_{in}(0)$: $0 = -\varepsilon(\alpha - 1) + e^{-(\alpha+1)}$

To leading order: $\varepsilon \alpha = e^{-\alpha}$

smaller $\sim O(\log|\log \varepsilon|)$

$$\log \varepsilon + \log \alpha = -\alpha$$

So $\alpha \sim -\log \varepsilon$ as $\alpha \to 0$.

Let $\alpha = -\log \varepsilon + b.$

$$\varepsilon(-\log \varepsilon + b - 1) = e^{-(-\log \varepsilon + b + 1)}$$

$$\log \varepsilon + \log(-\log \varepsilon + b - 1) = \log \varepsilon - (b+1)$$

$$\log\left[|\log \varepsilon|\left(1 + \frac{b-1}{|\log \varepsilon|}\right)\right] = -(b+1)$$

$$\underbrace{\log|\log \varepsilon|}_{\text{dominant term}} + \frac{b-1}{|\log \varepsilon|} + O\left(\frac{b}{|\log \varepsilon|^2}\right) = -(b+1)$$

So $b + 1 = -\log|\log \varepsilon|$

Check: $\dfrac{\log|\log \varepsilon|}{|\log \varepsilon|} \ll \log|\log \varepsilon|$ self-consistent

Conclude: $y_{in}(0) = \log \varepsilon + \log|\log \varepsilon| + 1 + \ldots$

For $\varepsilon = 0.01$, this is $\simeq -2.078$.

The numerical solution gives -2.942. slow convergence

This might seem like a big error, but the only other guess we had for $y(0)$ was $-\infty$!

example: $\varepsilon y'' + a(x)y' + b(x)y = 0 \quad 0 \lesssim x \lesssim 1$

$y(0) = A, \; y(1) = B$

Assume $a(x) \neq 0$, $0 \lesssim x \lesssim 1$ (see later), $a(x) > 0$

Otherwise a, b continuous.

Boundary layer as $x \to 0$.

applied $y_{out}(1) = B$, since this is in the outer region.

Outer: $a y'_{out} + b y_{out} = 0$

$$y_{out} = B \exp\left[\int_x^1 \frac{b(t)}{a(t)} dt\right]$$

$a \neq 0$, otherwise "internal layer"

Valid as $\varepsilon \to 0$ for $\delta \ll x \leq 1$.

If $y_{out}(0) = A$, we are done, but this is exceptional

Near $x = 0$, approximate $a(x) \simeq \alpha$, $b(x) \simeq \beta$.

Also, $y'/y \to \infty$ since y varies rapidly

Hence,

$$\varepsilon y''_{in} + \alpha y'_{in} = 0$$

$$y_{in} = A + C\left(e^{-\alpha x/\varepsilon} - 1\right)$$

Now for the asymptotic matching.

The boundary layer is of thickness $\delta \sim \varepsilon$

since $\dfrac{\varepsilon y''}{\alpha y'} \sim \dfrac{\varepsilon \delta^{-2}}{\delta^{-1}} \sim O(1) \Rightarrow \delta \sim \varepsilon$

Matching region: take $x \sim \varepsilon^{1/2}$, say $(\varepsilon \ll x \ll 1)$

$$y_{in}(x) \sim A - C$$

← matching depends on $\alpha > 0$

$$y_{out}(x) \sim B \exp\left[\int_0^1 b(t)/a(t)\, dt\right] = y_{out}(0)$$

Hence, $C = A - y_{out}(0)$, and

$$y(x) \sim B \exp\left[\int_0^1 b(t)/a(t)\, dt\right], \quad \varepsilon \ll x \lesssim 1$$

$$\sim A e^{-a(0)x/\varepsilon} + B\left(1 - e^{-a(0)x/\varepsilon}\right) \exp\left[\int_0^1 b(t)/a(t)\, dt\right]$$

$$x = O(\varepsilon)$$

Can combine using.

$$y_{unif} = y_{out} + y_{in} - \underbrace{y_{match}}_{A-C}$$

$$y_{unif} = B \exp\left[\int_x^1 b/a\, dt\right] + \left\{A - B \exp\left[\int_0^1 b/a\, dt\right]\right\} e^{-a(0)x/\varepsilon}$$

This is a "uniform" approximation in the sense that

$$|y(x) - y_{unif}(x)| \sim O(\varepsilon) \quad 0 \leq x \leq 1, \; \varepsilon \to 0^+$$

Observe: if $a(x) < 0$, boundary layer is at $y = 1$.