Lecture 21: Singular perturbation theory

For ODEs:

$$\varepsilon y'' - y' = 0, \quad y(0) = 0, \quad y(1) = 1$$

This is singular since the zeroth-order problem,

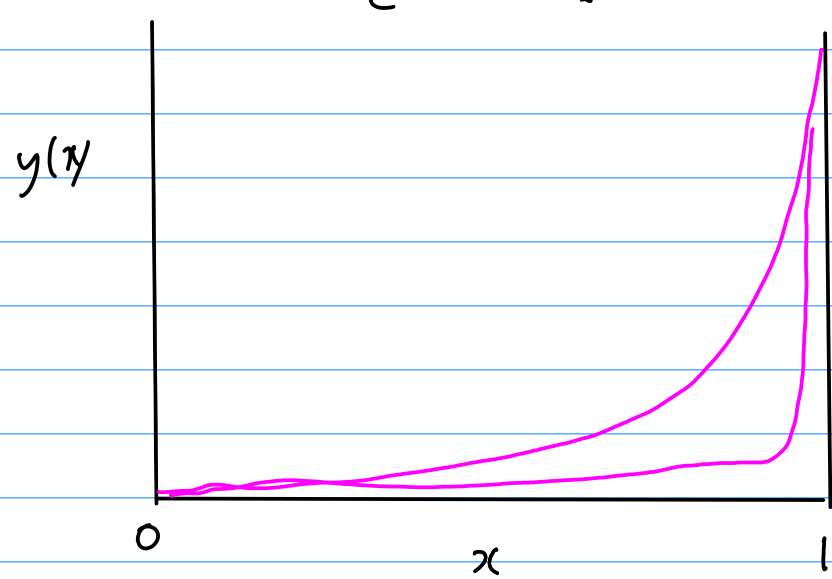
$$-y' = 0, \quad y(0) = 0, \quad y(1) = 1$$

has no solution! This often happens when $\varepsilon = 0$ lowers the order of the D.E.

In this case, the system has an exact solution which hints at the problem.

$$y(x) = \frac{e^{x/\varepsilon} - 1}{e^{1/\varepsilon} - 1}$$

$\lim_{\varepsilon \to 0}$ does not exist



solution has sharper "jump" as $\varepsilon \to 0$

BOUNDARY LAYER

width $\sim \varepsilon$

We will return to this later. [ASYMPTOTIC MATCHING]

Another singular example:

$$\varepsilon y'' + y = 0 \quad , \qquad y(0)=0, \; y(1)=1$$

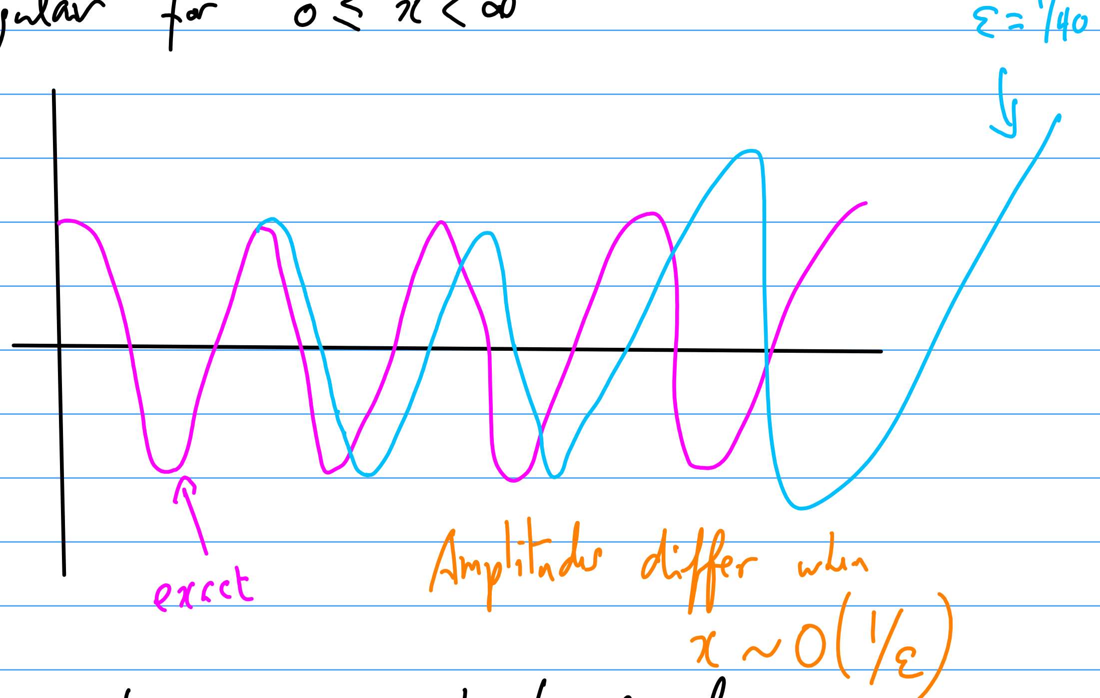
$\varepsilon = 0$ again has no solution.

Exact: $$y(x) = \frac{\sin\left(x/\sqrt{\varepsilon}\right)}{\sin\left(1/\sqrt{\varepsilon}\right)}$$ Fast oscillations [WKB Theory]

Yet another:

$$y'' + (1 - \varepsilon x) y = 0 \qquad \begin{array}{l} y(0) = 1 \\ y'(0) = 0 \end{array}$$

Regular over a finite interval $0 \leq x \leq L$, but singular for $0 \leq x < \infty$



This occurs because εx gets big for large x.

Asymptotic matching:

example: $y' + (\varepsilon x^2 + 1 + 1/x^2)\, y = 0 \qquad y(1) = 1$

$1 \leq x < \infty$

We can solve this exactly, but let's use pert. th. instead.

For moderate x, εx^2 is negligible, and

$y \sim y_L$, with $y_L' + \left(1 + \frac{1}{x^2}\right) y_L = 0 \qquad y_L(1) = 1$

"left" $\qquad\qquad \varepsilon x^2 \ll 1$

Solution: $y_L = e^{-x + 1/x}$

But for large x can no longer neglect εx^2.
However, $\frac{1}{x}$ is small!

$y \sim y_R, \qquad y_R' + (\varepsilon x^2 + 1)\, y_R = 0 \qquad \frac{1}{x^2} \ll 1$

$$y_R = a\, e^{-\varepsilon x^3/3 - x}, \quad a \text{ constant}$$

These solutions have a common region of validity:

$$y_L = e^{-x + \frac{1}{x}}, \qquad y_R = e^{-x + \frac{1}{3}\varepsilon x^3}$$

Can neglect $\frac{1}{x}$ vs x is $\frac{1}{x} \ll x \Rightarrow x^2 \gg 1$

Can neglect εx^3 vs x if $\varepsilon x^2 \ll 1$

So if $x \sim \varepsilon^{-1/n}$, $\varepsilon^{-2/n} \gg 1$, $\varepsilon^{1-\frac{2}{n}} \ll 1$

$n > 0$ $\qquad$ $1 - \frac{2}{n} > 0 \Rightarrow n > 2$

then both y_L and $y_R \sim e^{-x}$.

For instance, in the range

$$\varepsilon^{-1/5} \ll x \ll \varepsilon^{-1/4}$$

not unique!

We have $y_L \sim e^{-x}$, $y_R \sim a e^{-x}$.

But they must agree, so $a = 1$.

See exercise at the end for exact range of validity.

Let us carry the matching to higher order.

$$y_L(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \dots, \quad y_0(1) = 1$$

$y_1(1) = 0$

$y_2(1) = 0$

$\vdots$

$$y_0 = e^{-x + 1/x}$$

$$y_1' + \left(1 + \frac{1}{x^2}\right) y_1 = -x^2 y_0$$

$$\Rightarrow \quad y_1 = \left(\frac{1}{3} - \frac{x^3}{3}\right) e^{-x + 1/x}$$

Hence, $y_L = e^{-x + \frac{1}{x}} \left[1 + \underbrace{\frac{\varepsilon}{3}(1 - x^3)}_{\to 0 \text{ for } \varepsilon x^3 \to 0} + O(\varepsilon^2 x^6) \right]$

$\varepsilon^2 x^6 = (\varepsilon x^3)^2$

(answers prob. 7.30)

Now for large x: put $y = e^{S(x)}$

$$S' y + \left(\varepsilon x^2 + 1 + \frac{1}{x^2} \right) y = 0$$

$$S' + \varepsilon x^2 + 1 + \frac{1}{x^2} = 0$$

$$S = -\frac{1}{3} \varepsilon x^3 - x + \frac{1}{x} + c \quad \text{exact}$$

$$e^S = a e^{-x + \frac{1}{x}} e^{-\frac{1}{3}\varepsilon x^3} = a e^{-x + \frac{1}{x}} \left(1 - \frac{1}{3} \varepsilon x^3 + \dots \right)$$

in overlap region

So $a = 1 + \frac{\varepsilon}{3}$ (exact: $a = e^{\varepsilon/3}$)

Asymptotic matching can also be used to expand integrals.

example: $$I(x) = \int_0^{\pi/2} e^{ix\cos t}\,dt \quad , \quad x \gg 1$$

For large x, the Riemann-Lebesgue Lemma implies that $I(x) \to 0$ as $x \to \infty$. But how fast?

$\rightarrow$ method of stationary phase. $\varphi(t) = \cos t$

$\varphi'(t) = -\sin t$

$\varphi'(t) = 0$ for $t = 0$.

$$\varphi(t) = 1 - \frac{t^2}{2} + O(t^4)$$

$$I(x) \simeq e^{ix} \int_0^{\pi/2 \to \infty} e^{-\frac{i}{2}xt^2}\,dt = \sqrt{\frac{\pi}{2\left(\frac{i}{2}x\right)}}\, e^{ix}$$

$$= \sqrt{\frac{\pi}{2x}}\, e^{i(x - \pi/4)} \qquad \text{since } i^{-1/2} = e^{-i\pi/4}$$

This is the leading order in x.

We will do this more rigorously on next page.

Completely local: $\pi/2$ in the upper bound could be anything (finite).

To get next order, need global analysis