Lecture 20: Perturbation methods

Exploiting small parameters — the heart of AM.

example: $x^3 - 4.001x + 0.002 = 0$

Rewrite: $x^3 - (4+\varepsilon)x + 2\varepsilon = 0 \qquad \varepsilon = .001$

Assume solution is analytic in ε, for some complex disk centred on $\varepsilon = 0$:

$$x(\varepsilon) = \sum_{n=0}^{\infty} a_n \varepsilon^n = \sum_{n=0}^{\infty} x_n$$

Now solve order-by-order:

$\varepsilon = 0$: $\quad x^3 - 4x = 0$ with $x = a_0$

$a_0 = -2, 0, 2$ "zeroth-order" (chosen to be easy)

second-order approx

Now write $x = a_0 + a_1\varepsilon + a_2\varepsilon^2 + O(\varepsilon^3)$

Substitute in $x^3 - (4+\varepsilon)x + 2\varepsilon = 0$:

$$x^3 = a_0^3 + 3a_0^2 a_1 \varepsilon + 3a_0^2 a_2 \varepsilon^2 + 3a_1^2 a_0 \varepsilon^2 + O(\varepsilon^3)$$

$$(4+\varepsilon)x = 4(a_0 + a_1\varepsilon + a_2\varepsilon^2) + a_0\varepsilon + a_1\varepsilon^2 + O(\varepsilon^3)$$

Put together:

$$\underbrace{(a_0^3 - 4a_0)}_{0} + (3a_0^2 a_1 - 4a_1 - a_0 + 2)\varepsilon$$
$$+ (3a_0(a_0 a_2 + a_1^2) - 4a_2 - a_1)\varepsilon^2$$
$$+ O(\varepsilon^3)$$
$$= 0$$

must vanish separately!

Hence, $a_1(3a_0^2 - 4) = a_0 - 2 \Rightarrow a_1 = \dfrac{a_0 - 2}{3a_0^2 - 4}$

$$a_2(3a_0^2 - 4) = a_1 - 3a_0 a_1^2$$

$$\Rightarrow a_2 = \frac{a_1(1 - 3a_0 a_1)}{3a_0^2 - 4}$$

For $a_0 = -2$, $\quad a_1 = \dfrac{-4}{8} = \dfrac{-1}{2}$

$$a_2 = \frac{-\frac{1}{2}(1 - 3(-2)(-\frac{1}{2}))}{8} = \frac{1}{8}$$

So $\quad x_1 = -2 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + O(\varepsilon^3)$

$$x_2 = 0 + \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + O(\varepsilon^3)$$

$$x_3 = 2 + 0\varepsilon + 0\varepsilon^2 + O(\varepsilon^3)$$

all vanish, since $x = 2$ is an exact root!

In this case we can prove convergence for $|\varepsilon| < 1$.

Usually, if we can find a solution at order ε^0, can find the others.

example: $y'' = f(x) y$, $y(0) = 1$
$y'(0) = 1$
$f(x)$ continuous

Introduce ε such that $\varepsilon = 0$ is solvable:

$$y'' = \varepsilon f(x) y$$

$$y(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n(x)$$

with $y_0(0) = 1$, $y_0'(0) = 1$

$y_n(0) = 0$, $y_n'(0) = 0$, $n > 0$

ε^0: $\quad y_0'' = 0 \Rightarrow y_0 = 1 + x$

ε^n: $\quad y_n'' = f(x) y_{n-1}$, $y_n(0) = y_n'(0) = 0$, $n \geq 1$

sequence of inhomogeneous D.E.s.

Homogeneous D.E. is ε^0.

Solve by variation of parameters.

$$y_n(x) = \int_0^x dt \int_0^t ds\, f(s)\, y_{n-1}(s), \quad n \geq 1$$

Check: $y_n(0) = 0$, $y_n'(x) = \int_0^x ds\, f(s)\, y_{n-1}(s)$

$\Rightarrow y_n'(0) = 0$, $y_n''(x) = f(x)\, y_{n-1}(x)$.

Get iterative solutions

$$y_1(x) = \int_0^x dt \int_0^t ds\, (1+s) f(s)$$

$$y_2(x) = \int_0^x dt \int_0^t ds\, f(s) \int_0^s dv \int_0^v du\, (1+u) f(u)$$

$\vdots$

$$y(x) = 1 + x + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots + \varepsilon^N y_N(x) + \cdots$$

$$y_N(x) = \int_0^x dt_1 \int_0^{t_1} ds_1\, f(s_1) \int_0^{s_1} dt_2 \int_0^{t_2} ds_2\, f(s_2)$$

$$\cdots \int_0^{s_N} dt_N \int_0^{t_N} ds_N\, (1+s_N) f(s_N)$$

Assume $|f(t)| < K$ for $0 \leq |t| \leq |x|$

$$|y_N(x)| \leq K^N \left| \int_0^x dt_1 \int_0^{t_1} ds_1 \int_0^{s_1} dt_2 \int_0^{t_2} ds_2 \cdots \int_0^{s_N} dt_N \int_0^{t_N} ds_N \, (1+s_N) \right|$$

$$\leq K^N (1+|x|) \left| \int_0^x dt_1 \int_0^{t_1} ds_1 \int_0^{s_1} dt_2 \int_0^{t_2} ds_2 \cdots \int_0^{s_N} dt_N \int_0^{t_N} ds_N \right|$$

2N integrals

$$= K^N (1+|x|) \frac{x^{2N}}{(2N)!}$$

Hence, the series converges absolutely for all x.

These two examples are regular perturbation problems.

Why? Nonvanishing radius of convergence.

Approach unperturbed ($\varepsilon = 0$) solution smoothly as $|\varepsilon| \to 0$.

Singular example:

$$\varepsilon^2 x^6 - \varepsilon x^4 - x^3 + 8 = 0$$

$\varepsilon = 0$: $\quad x^3 = 8 \Rightarrow x = 2, 2\omega, 2\omega^2$

$$\omega = e^{2\pi i/3}$$

Problem: zeroth-order solution has 3 roots, but $\varepsilon \neq 0$ has 6!

$\longrightarrow$ singular problem

Why? The three missing roots $\to \infty$ as $\varepsilon \to 0$
Hence, $\varepsilon^2 x^6 - \varepsilon x^4$ is of the same order as $-x^3 + 8$

as $\varepsilon \to 0$.

For the other three roots there is no problem in using regular perturbation theory:

$$x_k(\varepsilon) = 2e^{2\pi i k/3} + \sum_{n=1}^{\infty} a_{n,k}\varepsilon^n, \quad k = 1, 2, 3$$

To find the other roots, estimate as $\varepsilon \to 0$

Four terms: $\varepsilon^2 x^6$, εx^4, x^3, 8

(1) (2) (3) (4)

So $\frac{4 \cdot 3}{2} = 6$ pairs. They balance as $\varepsilon \to 0$.

But how? Try all pairs.

X (a) $\underline{\varepsilon^2 x^6 \sim \varepsilon x^4}$: $x \sim \varepsilon^{-1/2}$, terms $\sim \varepsilon^{-1}$

Hence $x^3 \sim \varepsilon^{-3/2}$ is the largest term and is unbalanced

X (b) $\underline{\varepsilon x^4 \sim x^3}$: $x \sim \varepsilon^{-1}$, so $\varepsilon^2 x^6 \sim \varepsilon^{-4}$ unbalanced

X (c) $\underline{\varepsilon^2 x^6 \sim 8}$: $x \sim \varepsilon^{-1/3}$, so $\varepsilon x^4 \sim \varepsilon^{-1/3}$ is ok, but $x^3 \sim \varepsilon^{-1}$ unbalanced

X (d) $\underline{\varepsilon x^4 \sim 8}$: $x \sim \varepsilon^{-1/4}$, $\varepsilon^2 x^6 \sim \varepsilon^{1/2}$ is ok but $x^3 \sim \varepsilon^{-3/4}$ is unbalanced

(e) $\underline{x^3 \sim 8}$: regular case

(f) $\underline{\varepsilon^2 x^6 \sim x^3}$: $x \sim \varepsilon^{-2/3}$, so $\varepsilon x^4 \sim \varepsilon^{-5/3} \ll \varepsilon^{-2}$

This case is ok! $\varepsilon^2 x^6$ and x^3 balance each other.

Make a scale transformation:

$$x = \varepsilon^{-2/3} y .$$

note: regular if $y \sim \varepsilon^{2/3}$

smallest power of ε

Then:

$$y^6 - y^3 + 8\varepsilon^2 - \varepsilon^{1/3} y^4 = 0$$

This is now a regular problem for our three missing roots!

$\varepsilon = 0$: $y^6 - y^3 = 0$: $y = 1, w, w^2, 0, 0, 0$

No roots disappear as $\varepsilon \to 0$.

$$y = \sum_{n=0}^{\infty} y_n \cdot (\varepsilon^{1/3})^n$$

cannot match coeffs for integral powers

For $y_0 = 0$, find $y_1 = 0$ and $y_2 = 2, 2w, 2w^2$

So $x = \varepsilon^{-2/3}(y_0 + \varepsilon^{1/3} y_1 + \varepsilon^{2/3} y_2 + \cdots)$

$= y_2 + \cdots$ becomes a regular expansion in ε again

But for $y_0 = 1, w, w^2$ get $y_1 \neq 0$, so expansion in $\varepsilon^{1/3}$.