Lecture 19: Chaos part 2 – flows

Last lecture: period-doubling cascade → chaos

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \lim_{\delta\to 0} \log \left| \frac{x(x_0+\delta) - x(x_0)}{\delta} \right|$$

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \log \left| x'(x_0) \right| \quad \text{Lyapunov exponent}$$

"tangent map"

There are other "routes" to chaos: - intermittency
- homoclinic chaos

Flows: (ODEs) $\dot{x}_1 = f(x_1, x_2)$, $\dot{x}_2 = g(x_1, x_2)$

not chaotic by Poincaré–Bendixson, but perhaps on the torus?

Take: $\dot{x}_1 = 2x_1 + x_2$, $\dot{x}_2 = x_1 + x_2$, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in T^2$ ← 2-torus

$T^2 = [0,1)^2$, bi-periodic

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \bmod 1$$

Is this chaotic? "Arnold's Cat Flow"

Write a solution as $x(x_0, t) = \begin{pmatrix} x_1(x_0, t) \\ x_2(x_0, t) \end{pmatrix}$

Distance between two trajectories:

$$x(x_0+\delta, t) - x(x_0, t) = \sum_j \frac{\partial x}{\partial x_{0_j}}(x_0, t)\, \delta_j + o(\delta^2)$$

$$\frac{x(x_0+\delta, t) - x(x_0, t)}{\|\delta\|} = \sum_j \frac{\partial x}{\partial x_{0_j}}(x_0, t)\, \frac{\delta_j}{\|\delta\|} + o(\delta)$$

neglect

unit vector $\hat{\delta}$

Write: $[Dx(x_0, t)]_{ij} = \frac{\partial x_i}{\partial x_{0_j}}(x_0, t)$

tangent map

Lyapunov exponent:

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \log \| Dx(x_0, t)\, \hat{\delta} \|$$

initial condition

initial direction of separation

For $\dot{x} = F(x, t)$, with solution $x(x_0, t)$,

$$\frac{d}{dt}\left(x(x_0+\delta, t) - x(x_0, t)\right) = F(x(x_0+\delta, t), t) - F(x(x_0, t), t)$$

$$= F(x(x_0, t) + Dx\,\delta, t) - F(x, t) + o(\delta^2)$$

$$= \frac{\partial F}{\partial x}(x, t)\, Dx\, \delta + (\text{neglect})$$

$$\frac{d}{dt}(Dx\,\hat{\delta})_i = \sum_j \frac{\partial F_i}{\partial x_j}(Dx\,\hat{\delta})_j$$

For Arnold's cat flow, $\frac{\partial F_i}{\partial x_j} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

So: $$Dx\,\hat{\delta} = \exp\left(\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} t\right)\hat{\delta}$$

Golden ratio

$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is diagonalizable, with eigenvalues

$$\lambda_\pm = \frac{1}{2}\left(3 \pm \sqrt{5}\right) = \varphi^2$$

$\hat{\delta}$ will typically contain a mixture of unstable/stable eigenvectors: $\hat{u}, \hat{s}$

$$\hat{\delta} = a\,\hat{u} + b\,\hat{s}$$

$$e^{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}t}\,\hat{\delta} = a\,e^{\lambda_+ t}\,\hat{u} + b\,e^{\lambda_- t}\,\hat{s}$$

Hence: $\|Dx\,\hat{\delta}\| \sim a\,e^{\lambda_+ t}$ for $t \gg 1$

So $$\boxed{\lambda = \lambda_+}$$

This system is chaotic because of the periodic boundary conditions

Homoclinic chaos:

Hamiltonian system: $\dot{x} = \frac{\partial H}{\partial y}$ $\quad H(x,y)$

$$\dot{y} = -\frac{\partial H}{\partial x}$$

$H(x,y)$ is constant: $\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = 0$
(on trajectories)

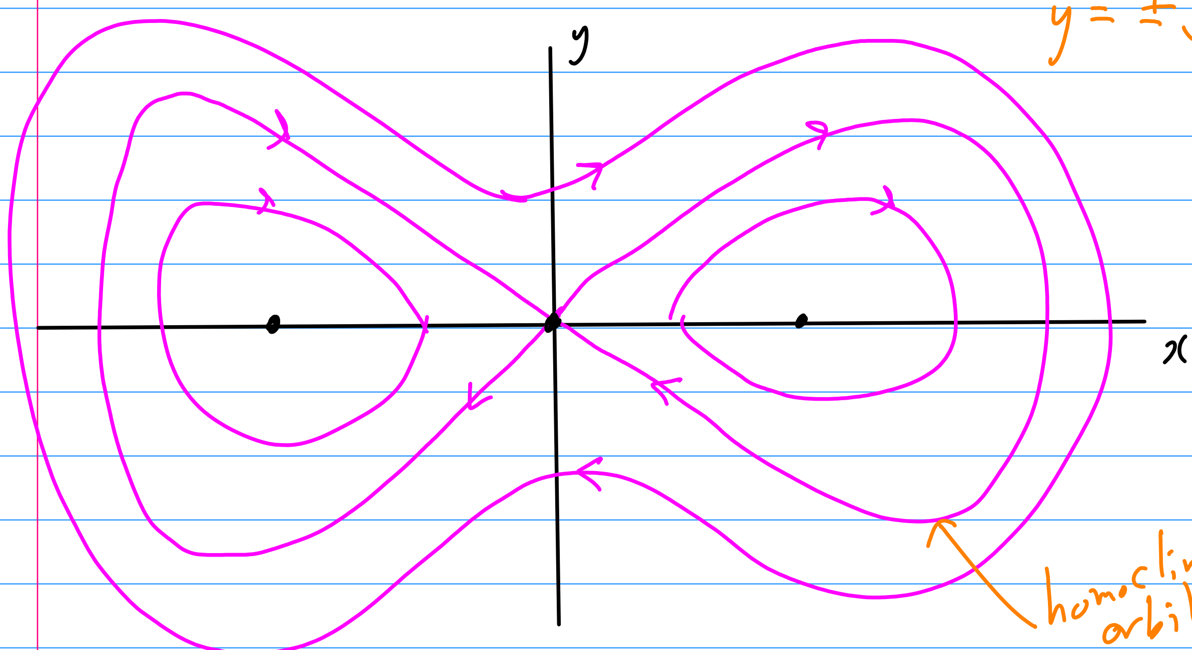
This means trajectories are contours of H!

Consider:

$$H = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$$

$$\dot{x} = y \; , \quad \dot{y} = x - x^3$$

$y = \pm\sqrt{x^2 - \frac{1}{2}x^4}$



homoclinic orbit

How does a Hamiltonian system respond to small perturbations?

periodic in time

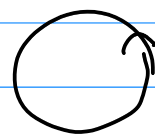
$$\dot{x} = \frac{\partial}{\partial y}\left(H + \varepsilon(x,y,t)\right)$$

$$\dot{y} = -\frac{\partial}{\partial x}\left(H + \varepsilon(x,y,t)\right)$$

"2+1 dimensions" since explicitly time-dep.

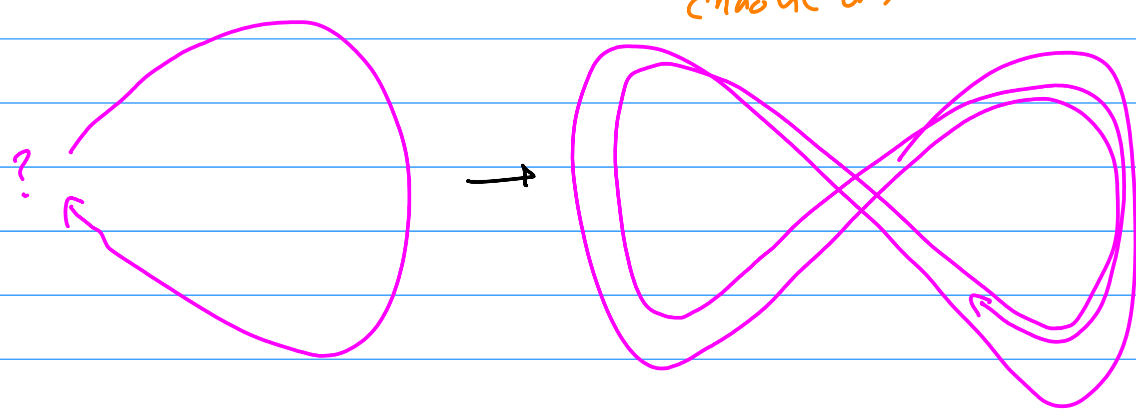
This is the topic of KAM theory.

(Kolmogorov-Arnold-Moser)

The closed orbits  are subject to resonance and might get destroyed. Whether they do or not depends on Diophantine conditions.

The most fragile orbits are the homoclinic/heteroclinic ones. Why? There is no fixed point to be connected to anymore!

chaotic orbit

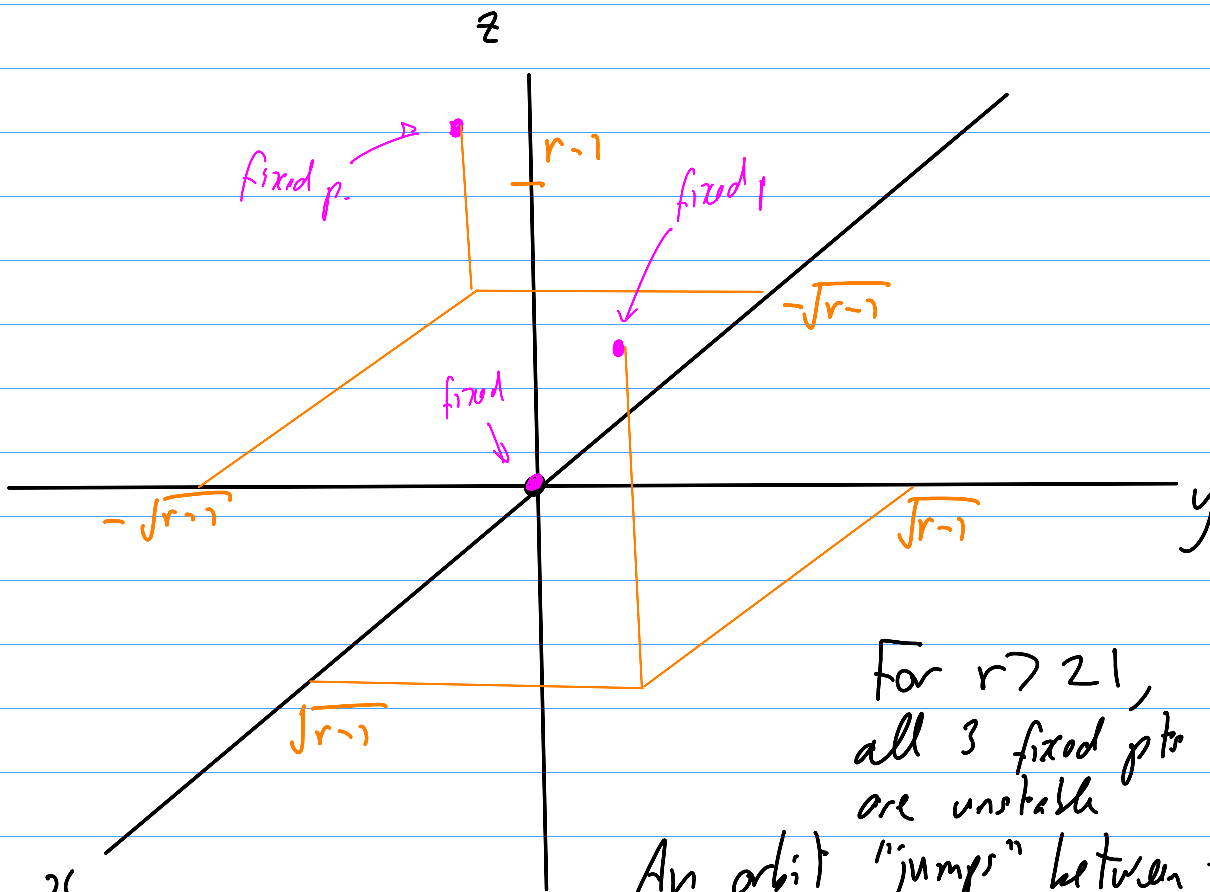


Lorenz equations: 3-dimensional, autonomous chaotic system

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - z$$

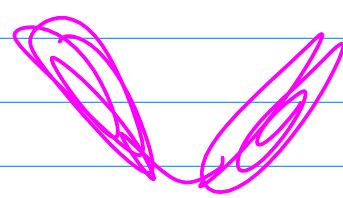
Take $\sigma = 3$

Fixed pts at
$(x,y,z) = (0,0,0)$
$(\pm\sqrt{r-1}, \pm\sqrt{r-1}, r-1)$



For $r > 21$,
all 3 fixed pts
are unstable

An orbit "jumps" between them.



Creates famous "butterfly" picture

"strange attractor"

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