Lecture 18: Chaos part 1: maps

Poincaré–Bendixson theorem: $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$

D closed bounded region of the x-y plane.

f, g continuously differentiable in D.

If a trajectory remains in D for all $t \geq 0$, then as $t \to \infty$ the trajectory must

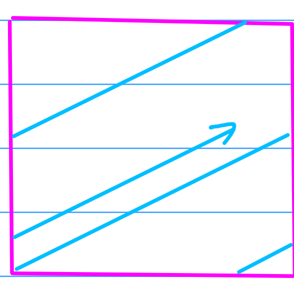
1) be a fixed point or closed orbit or

2) approach a fixed point or closed orbit.

See link on website for a nice discussion/proof.

This is not true in other types of region:
for example, consider the torus: $T = [0,1]^2$, periodic

$\dot{x} = 1$
$\dot{y} = \alpha$



For α irrational, an orbit will never close.

(But we will not call this "chaotic")

So what do we need for chaos?

(We assume motion in a bounded region.)

Need: • non-autonomous system ("2+1")

• ≥ 3 dimensions

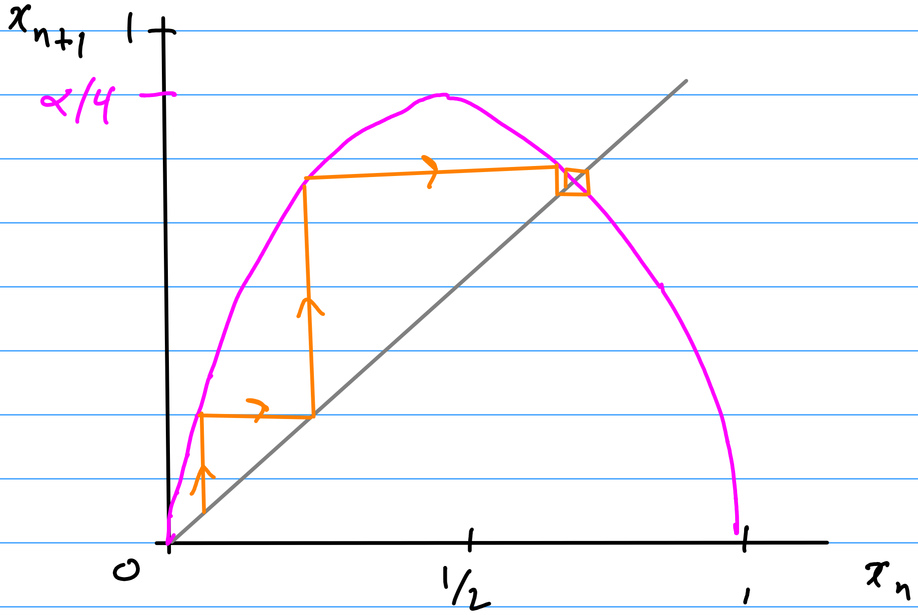
• A mapping

Mappings are the easiest way to get chaos:

$$x_{n+1} = f(x_n) \qquad \text{Can be chaotic even in 1D.}$$

example: logistic map

$$x_{n+1} = \alpha x_n (1 - x_n)$$



In 1D maps, fixed points are at the intersection of the line $y = x$ and the curve $y = f(x)$.

$$f(x_*) = x_* \quad \text{fixed point}$$

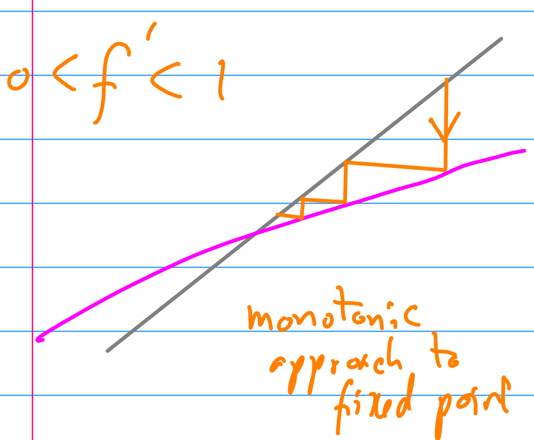
Stability of fixed point: $x = x_* + \varepsilon$

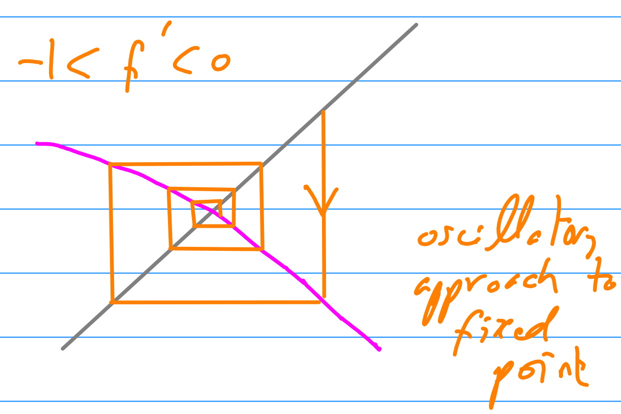
$$x_* + \varepsilon_{n+1} = f(x_* + \varepsilon_n)$$

$$= f(x_*) + f'(x_*)\,\varepsilon_n + O(\varepsilon_n^2)$$

$$\Rightarrow \quad \varepsilon_{n+1} = f'(x_*)\,\varepsilon_n \qquad 0 \le f' < 1 : \text{monotonic}$$

Unstable if $|f'(x_*)| > 1$. $\qquad -1 < f' < 0 : \text{oscillation}$





For logistic map, two fixed points: $x_* = 0$ and

$$x_* = \alpha\, x_* (1 - x_*) \Rightarrow x_* = 1 - \alpha^{-1}$$
$$(x_* \neq 0)$$

Stability: $f'(x) = \frac{d}{dx}\left[\alpha x(1-x)\right] = \alpha(1-2x)$

$f'(0) = \alpha$ stable for $|\alpha| < 1$

$f'(1-\alpha^{-1}) = \alpha(1-2(1-\alpha^{-1})) = 2-\alpha$

stable for $|2-\alpha| < 1$

For $0 < \alpha < 1$, $x_* = 0$ is stable, but $x_* = 1-\alpha^{-1}$ unstable

For $1 < \alpha < 3$, $x_* = 0$ is unstable, but $x_* = 1-\alpha^{-1}$ is stable

For $\alpha > 3$ both are unstable.

Now we can get periodic orbits: $x_{n+2} = x_n$ (period 2)

$$x_{n+2} = f(x_{n+1}) = \alpha x_{n+1}(1-x_{n+1})$$
$$= \alpha x_n(1-x_n)(1-\alpha x_n(1-x_n))$$

Solve $= x_n$ for period-2 orbits

$$x = \alpha x(1-x)(1-\alpha x(1-x))$$

$x = 0$ is still a solution, as is $x = 1-\alpha^{-1}$

(the fixed points)

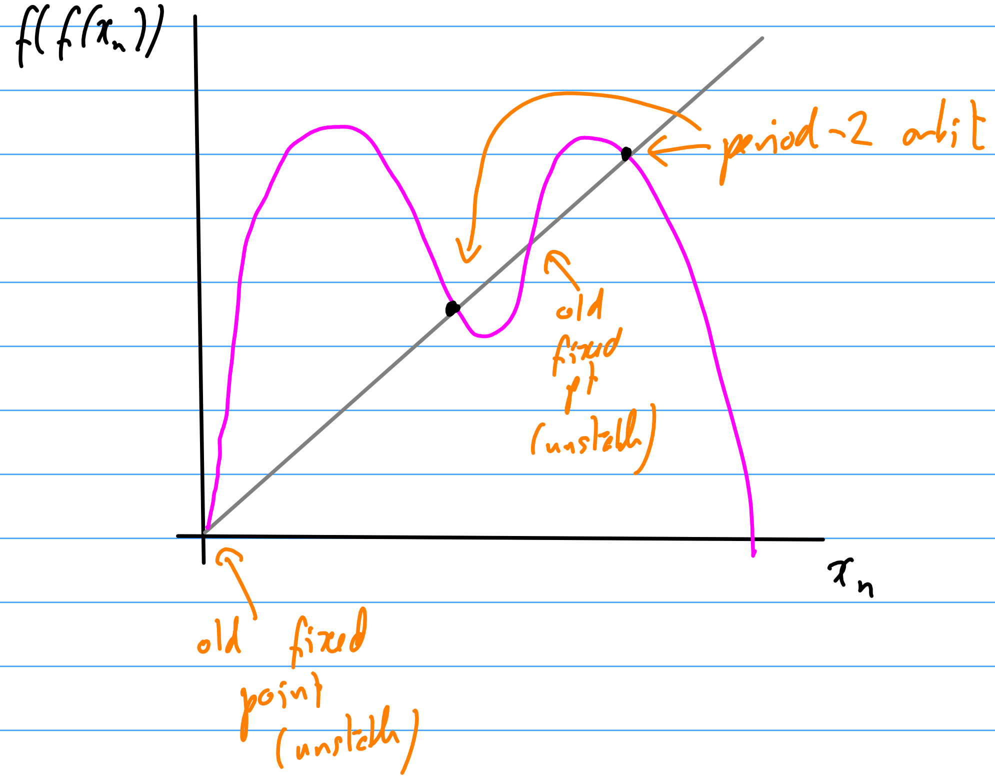
Two more solutions:

$$x_{\pm} = \frac{1}{2}\left(1 + \alpha^{-1} \pm \sqrt{(1+\alpha^{-1})(1-3\alpha^{-1})}\right)$$

These are the two points of the period-2 orbit!

This orbit won't exist unless $1 - 3\alpha^{-1} > 0 \Rightarrow \alpha > 3$

But that coincides with both fixed points losing stability.



This scenario repeats itself as we increase α:

period 2 becomes unstable → period 4 becomes stable, etc.

"period-doubling bifurcations"

But the surprising thing is that these bifurcations occur an infinite # of times in a finite interval of α.

"period-doubling cascade"

$\rightarrow$ CHAOS

$\alpha = 4$ is particularly interesting.

Consider a "density" of points $\rho(x) \geq 0$, $x \in [0,1]$.

How is a density mapped forward?

$$\rho_{n+1}(x) = \int_0^1 \rho_n(y)\, \delta(x - f(y))\, dy$$

all the points now at x come from y

$x = f(y)$ has two solutions: $y_\pm = \frac{1}{2}(1 \pm \sqrt{1-x})$

$$f'(y_\pm) = 4(1-2y_\pm) = 4\left(1 - 2\tfrac{1}{2}(1 \pm \sqrt{1-x})\right)$$

$$= 4(\mp\sqrt{1-x}) = \mp 4\sqrt{1-x}$$

$$\rho_{n+1}(x) = \frac{\rho_n(y_+)}{|f'(y_+)|} + \frac{\rho_n(y_-)}{|f'(y_-)|}$$

$$\delta(x - f(y)) = \sum \frac{\delta(y - y_\pm)}{|f'(y_\pm)|}$$

Note that $1 - y_- = 1 - \frac{1}{2}(1 - \sqrt{1-x}) = y_+$

Assume $\rho_n(1-x) = \rho_n(x)$ (symmetric)

$$\rho_n(y_-) = \rho_n(1-y_+) = \rho_n(y_+)$$

Hence,

$$\rho_{n+1}(x) = 2\,\frac{\rho_n(y_+(x))}{|f'(y_+(x))|} = \frac{\rho_n(y_+(x))}{2\sqrt{1-x}}$$

Can we find the "invariant distribution" $\rho(x)$?

$$\rho(x) = \frac{\rho\left(\frac{1}{2}(1+\sqrt{1-x})\right)}{2\sqrt{1-x}}$$

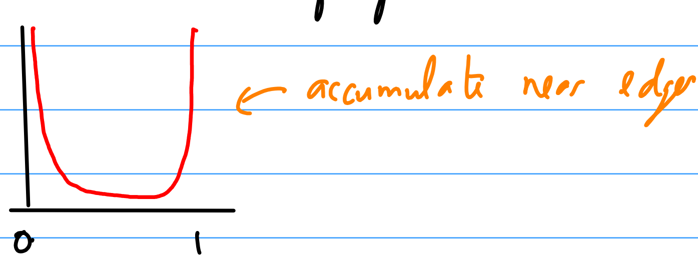
Check: $\rho(x) = \frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}$ satisfies this.

$$\rho\left(\frac{1}{2}(1+\sqrt{1-x})\right) = \frac{1}{\pi}\left(\frac{1}{2}(1+\sqrt{1-x})\,\frac{1}{2}(1-\sqrt{1-x})\right)^{-1/2}$$

$$= \frac{1}{\pi}\,2\,\left(1-(1-x)\right)^{-1/2} = \frac{2}{\pi}\frac{1}{\sqrt{x}}$$

$$= 2\sqrt{1-x}\,\rho(x)$$

What does this mean? This invariant distribution tells us how a cloud of points becomes distributed on [0,1]



For $\alpha = 4$, the map itself also has an exact solution:

$$x_{n+1} = 4x_n(1-x_n)$$

$$\Rightarrow \quad x_n = \sin^2(2^n \theta(x_0))$$

where $x_0 = \sin^2(\theta(x_0))$, so $\theta(x_0) = \sin^{-1}\sqrt{x_0}$

Consider two "nearby" trajectories:

starting at $\theta(x_0)$, $\theta(x_0+\delta)$

$$x_n(x_0+\delta) - x_n(x_0) = 2^{n-1} \frac{\sin(2^{n+1}\theta(x_0))}{\sqrt{x_0(1-x_0)}}\delta + O(\delta^2)$$

Define:

$$\lambda = \lim_{n\to\infty} \lim_{\delta\to 0} \frac{1}{n} \log\left|\frac{x_n(x_0+\delta) - x_n(x_0)}{\delta}\right|$$

derivative as $\delta \to 0$

$$\boxed{\lambda = \lim_{n\to\infty} \frac{1}{n} \log|x_n'(x_0)|}$$

"asymptotic rate of separation" Lyapunov exponent

In this case $\lambda = \log 2$

$\underline{\lambda > 0 \text{ defines chaos}}$

```
%
% Example program for invariant distribution and Lyapunov exponent of
% logistic map
%
% Jean-Luc Thiffeault (jeanluc@math.wisc.edu)
%

Npts = 20000; Nit = 50; nbins = 50; r = 4;
x = rand(1,Npts);        % random initial points
f = @(r,x) r.*x.*(1-x); % logistic map

figure(1)

% the initial points are uniformly distributed on the interval [0,1]
subplot(2,1,1)
[P,xb] = hist(x,nbins); P = P./trapz(xb,P);
bar(xb,P,1)
title('initial distribution')

% now iterate the map on the initial points
for i = 1:Nit, x = f(r,x); end

% the final points obey the invariant distribution
subplot(2,1,2)
[P,xb] = hist(x,nbins); P = P./trapz(xb,P);
bar(xb,P,1)
title(sprintf('distribution after %d iterations',Nit))
if r == 4
  hold on
  xi = linspace(0,1,300); xi = xi(2:end-1);
  plot(xi,1/pi*(xi.*(1-xi)).^-.5,'r','LineWidth',2) % analytic answer for r=
  hold off
end

figure(2)

% two trajectories very close to each other initially
delta = 1e-10; x1 = [.1]; x2 = [x1 + delta];

% iterate the map on both trajectories
for i = 1:Nit, x1 = [x1 f(r,x1(end))]; x2 = [x2 f(r,x2(end))]; end
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% plot the separation
semilogy(0:Nit,abs(x1-x2))
hold on
semilogy(0:Nit,delta*2.^(0:Nit),'r') % compare to 2^n separation rate
hold off
axis([0 Nit delta 10])
xlabel('n')
ylabel('distance between trajectories')

figure(3)

Nit = 2000; Nplot = 1000; r = linspace(0,4,400); x = .1*ones(size(r));

% iterate the map on the same initial condition, but different r
for i = 1:Nit, x = [x; f(r,x(end,:))]; end

clf, hold on
% plot the last Nplot iterates for each r
for i = 1:Nplot
  plot(r,x(Nit-i-1,:),'.','MarkerSize',2)
end
hold off
xlabel('r')
title('bifurcation diagram for logistic map')
```

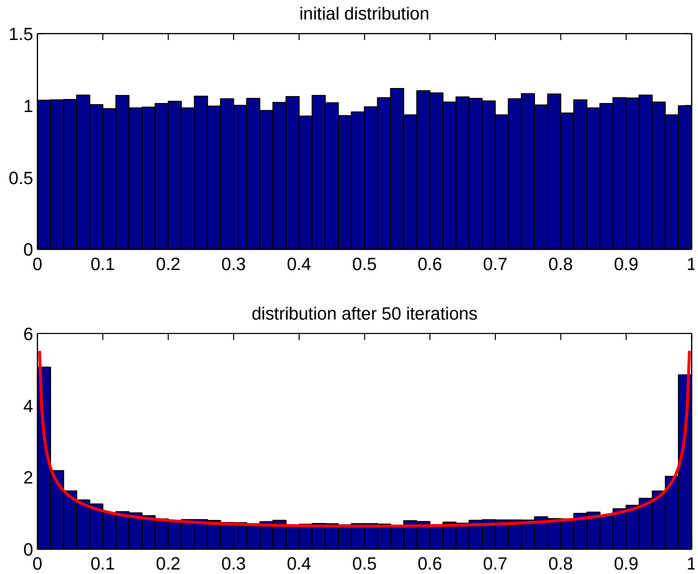

Figure 1: Invariant distribution for the logistic map with $r = 4$.

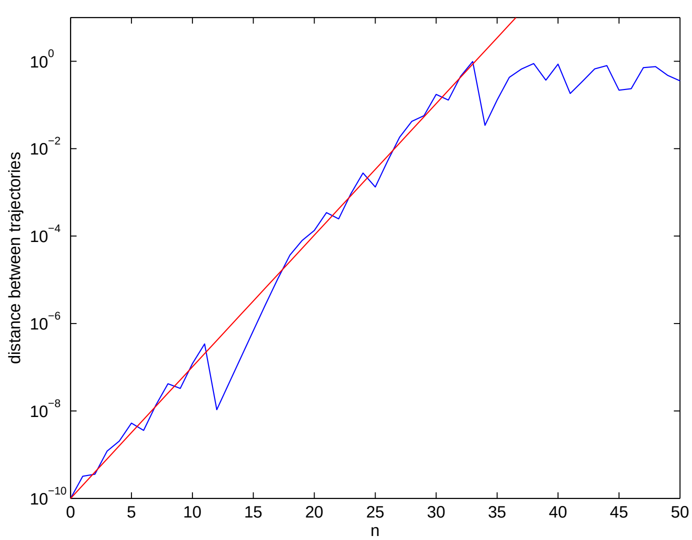

Figure 2: Separation of nearby trajectories for the logistic map with $r = 4$. The red line is 2^n.

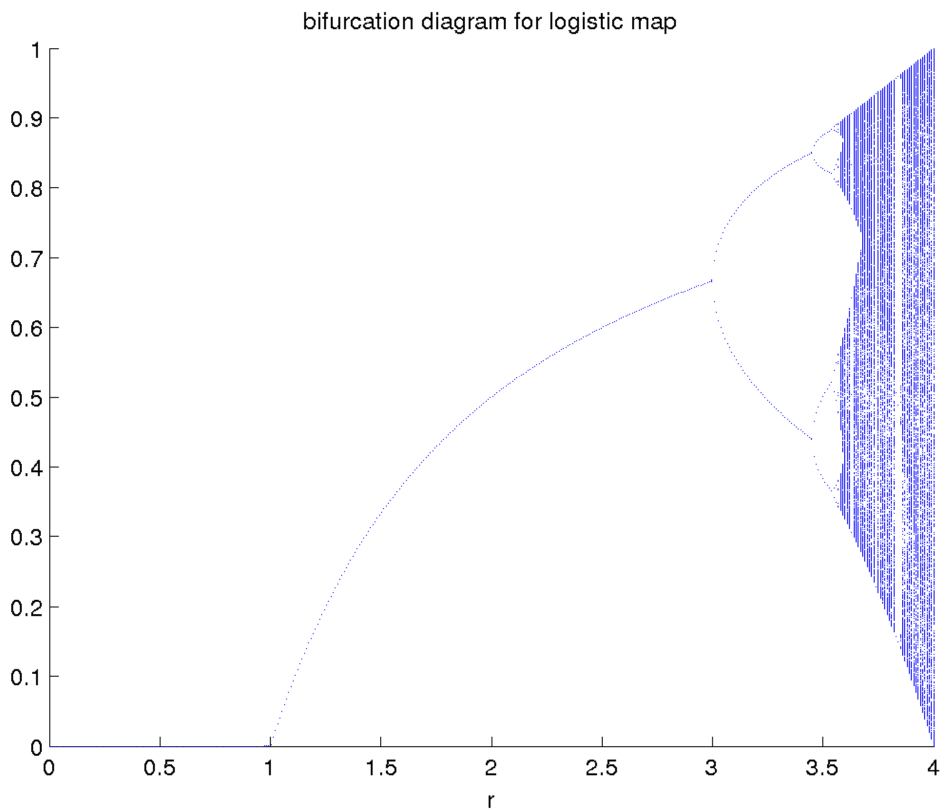

Figure 3: Bifurcation diagram for the logistic map. For each r, the map is iterated 2000 times starting from $x = 0.1$, and the next 1000 iterates are plotted.