Lecture 17: Phase plane analysis

$$\dot{y} = F(y) \qquad y \in \mathbb{R}^2$$

How to analyze this system?

First find critical (fixed) points: $F(y) = 0$

Then examine linear stability of each point.

saturation predator birth/death

example:

$$\dot{y}_1 = y_1(3 - y_1 - y_2) = -y_1^2 - y_1 y_2 + 3y_1$$

$$\dot{y}_2 = y_2(y_1 - 1) = +y_1 y_2 - y_2$$

predator-prey

Fixed point, $y_2 = 0 \Rightarrow y_1 = \begin{cases} 0 \\ 3 \end{cases}$

$$y_1 = 1 \Rightarrow y_2 = 2$$

So 3 critical points: 1 $(0,0)$, 2 $(1,2)$, 3 $(3,0)$

Now linearize near each point: $y = y_* + \varepsilon$

1 $\dot{\varepsilon}_1 = 3\varepsilon_1$, $\dot{\varepsilon}_2 = -\varepsilon_2$

$$A = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

saddle $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ unstable $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ stable

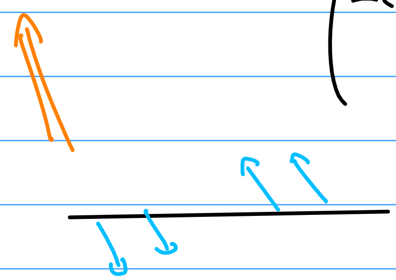
(2) $$\dot{\varepsilon}_1 = (1+\varepsilon_1)(\cancel{3} - (\cancel{1}+\varepsilon_1) - (\cancel{2}+\varepsilon_2))$$
$$= -\varepsilon_1 - \varepsilon_2 + O(\varepsilon^2)$$
$$\dot{\varepsilon}_2 = (2+\varepsilon_2)((\cancel{1}+\varepsilon_1) - \cancel{1}) = 2\varepsilon_1 + O(\varepsilon^2)$$

$$A = \begin{pmatrix} -1 & -1 \\ 2 & 0 \end{pmatrix} \qquad \begin{aligned} \operatorname{tr} A &= -1 \\ \det A &= 2 \end{aligned}$$

$(\operatorname{tr} A)^2 - 4(\det)A = -7 < 0$
stable spiral

Is it  or  ? Check intersection with ε_1 axis:

$$\left.\begin{pmatrix} -\varepsilon_1 - \varepsilon_2 \\ 2\varepsilon_1 \end{pmatrix}\right|_{\varepsilon_2 = 0} = \begin{pmatrix} -\varepsilon_1 \\ 2\varepsilon_1 \end{pmatrix}$$

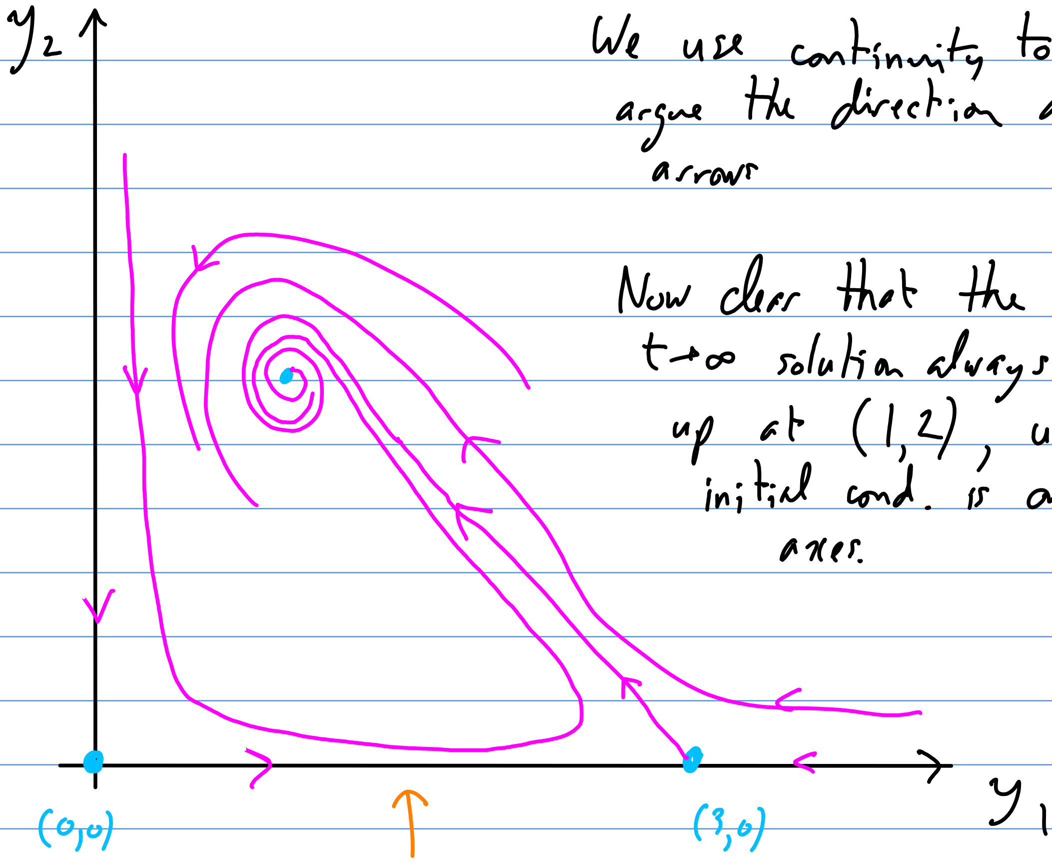


(3) $$\dot{\varepsilon}_1 = (3+\varepsilon_1)(\cancel{3} - (\cancel{3}+\varepsilon_1) - \varepsilon_2) = (3+\varepsilon_1)(-\varepsilon_1 - \varepsilon_2) + O(\varepsilon^2)$$
$$= -3\varepsilon_1 - 3\varepsilon_2 + O(\varepsilon^2)$$
$$\dot{\varepsilon}_2 = \varepsilon_2((3+\varepsilon_1) - 1) = 2\varepsilon_2 + O(\varepsilon^2)$$

$$A = \begin{pmatrix} -3 & -3 \\ 0 & 2 \end{pmatrix} \qquad \begin{aligned} \operatorname{tr} A &= -1 \\ \det A &= -6 \end{aligned}$$

saddle
$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$
stable, unstable

Now comes the fun part: we must assemble these fixed points in the phase plane, and connect them appropriately.



We use continuity to argue the direction of arrows

Now clear that the $t \to \infty$ solution always ends up at $(1,2)$, unless initial cond. is on the axes.

heteroclinic orbit (connects two fixed points)

Another example: B&O p.181

$$\dot{y}_1 = y_1^2 - y_1 y_2 - y_1$$
$$\dot{y}_2 = y_2^2 + y_1 y_2 - 2y_2$$

reproduction, predation, death

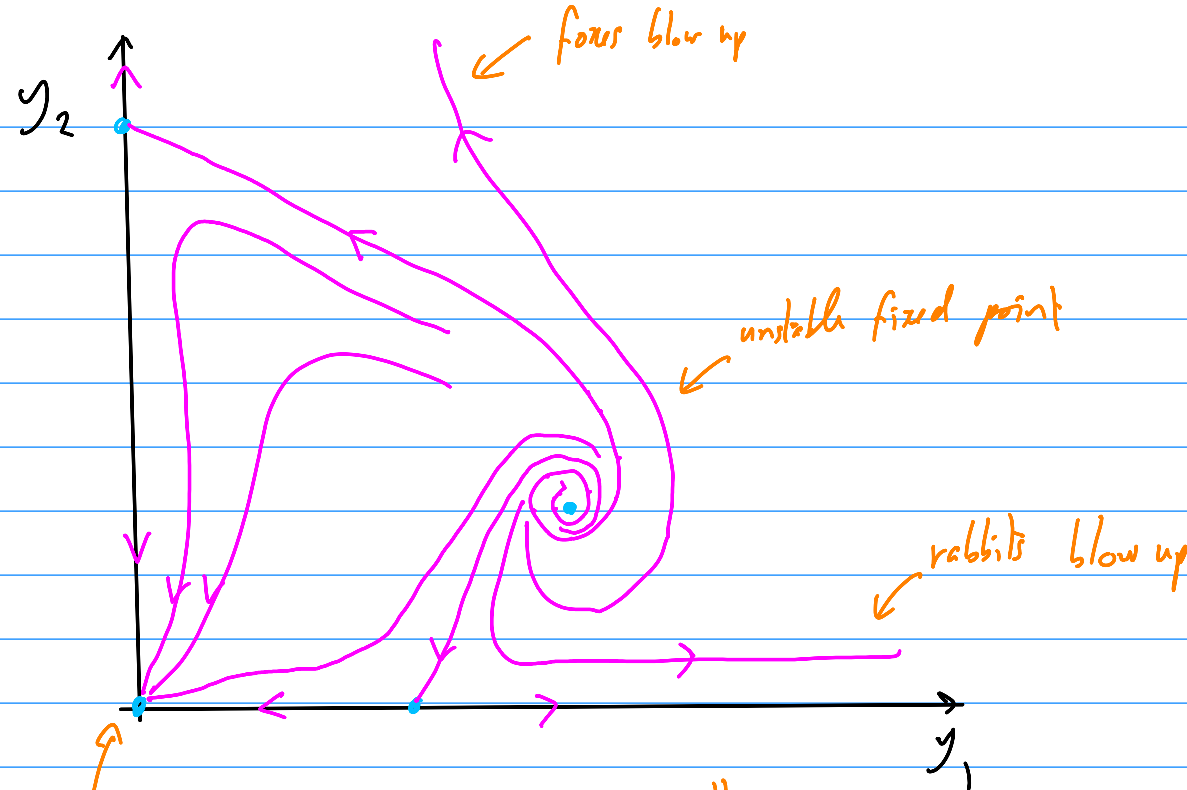
predator-prey with reproduction

y_1 = rabbits

y_2 = foxes

This has 4 critical points

$$(0,0),\ (1,0),\ (3/2,\ 1/2),\ (0,2)$$



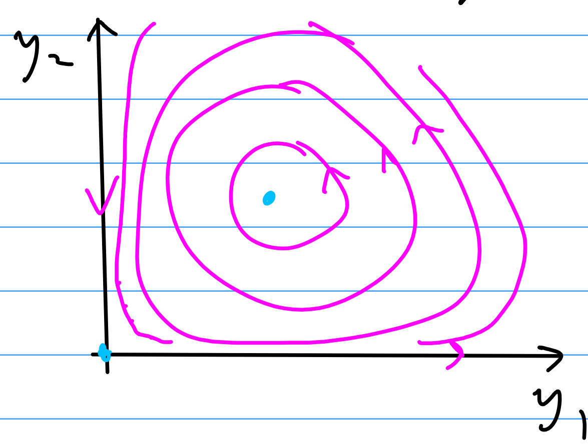
birth/death

predator

Lotka-Volterra: $\dot{y}_1 = y_1 - y_1 y_2$

$\dot{y}_2 = -y_2 + y_1 y_2$

Two fixed pts $(0,0)$, $(1,1)$



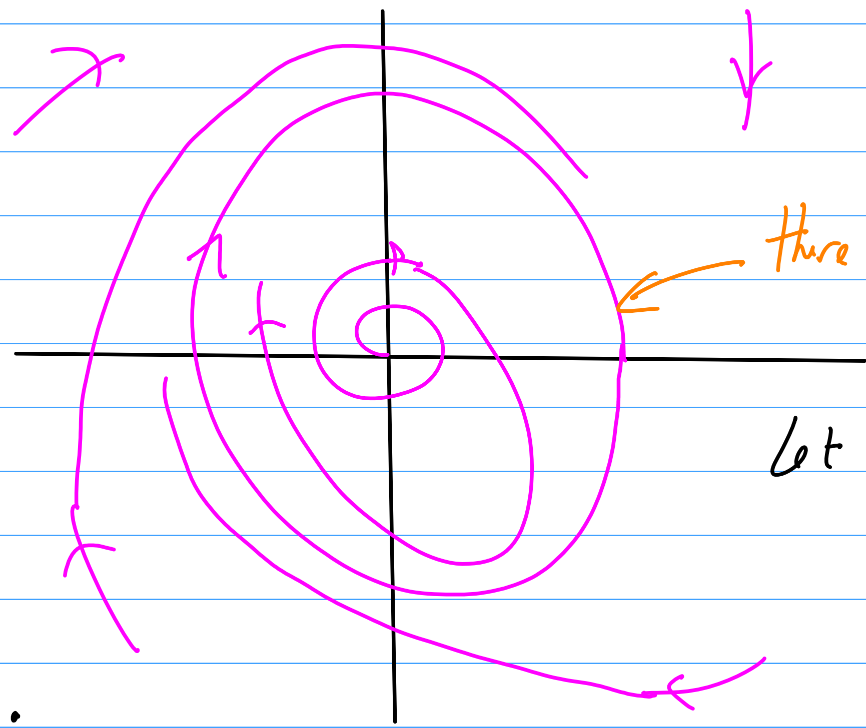
Only closed orbits for $y_1 > 0$, $y_2 > 0$.

<u>example:</u>

$$\dot{y}_1 = y_1 + y_2 - y_1(y_1^2 + y_2^2)$$
$$\dot{y}_2 = -y_1 + y_2 - y_2(y_1^2 + y_2^2)$$

Unstable spiral at $(0,0)$. No other fixed points.

For large $y_1^2 + y_2^2$, $\dot{y}_1 < 0$, $\dot{y}_2 < 0$ inward!



There must exist a limit cycle!

Let $r^2 = y_1^2 + y_2^2$

$$(r^2)^{\cdot} = 2y_1\dot{y}_1 + 2y_2\dot{y}_2$$
$$= 2(y_1^2 + \cancel{y_1 y_2} - y_1^2 r^2) + 2(\cancel{-y_1 y_2} + y_2^2 - y_2^2 r^2)$$
$$= 2(r^2 - r^4)$$

So $\dot{r} = 0$ for $r = 1$!

This is the limit cycle.

This "local analysis" doesn't always work:

$$\dot{y}_1 = y_1^2 + y_2^2 + y_1^6$$
$$\dot{y}_2 = \sin(y_1^4 + y_2^4)$$

Near $(0,0)$, $\dot{\varepsilon}_1 = \varepsilon_1^2 + \varepsilon_2^2 + (\text{smaller})$

$$\dot{\varepsilon}_2 = \varepsilon_1^4 + \varepsilon_2^4 + (\text{smaller})$$

So if we neglect nonlinear terms we get $(\varepsilon_1, \varepsilon_2) = \text{const.}$, so we can't determine stability.

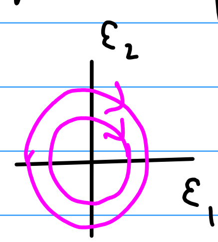
Much harder to analyze.

<u>example:</u> $\dot{y}_1 = -y_2 + y_1(y_1^2 + y_2^2)$

$$\dot{y}_2 = y_1 + y_2(y_1^2 + y_2^2)$$

center

$(0,0)$ is the only critical point. $\begin{pmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$



Let's do this another way:

let $r^2 = y_1^2 + y_2^2$

$$(r^2)^{\cdot} = 2(y_1\dot{y}_1 + y_2\dot{y}_2) = 2y_1^2(y_1^2+y_2^2) + 2y_2^2(y_1^2+y_2^2)$$

$$\frac{1}{2}(r^2)^{\cdot} = r^4 > 0 \quad \text{for } r > 0$$

So the origin is an unstable spiral.

$r \to \infty$ in finite time $\quad r(t) = \dfrac{r(0)}{\sqrt{1 - 2r^2(0)t}}$

We say that centers are not "structurally stable": any small perturbation can destroy a closed orbit, since it depends on "catching its tail"