Lecture 16: Stability

autonomous system

$$u' = F(u) \qquad u \in \mathbb{R}^n$$

Take a reference solution $u_*(t)$

Is it "stable"? That is, do initial values "near" $u_*(t)$ remain close?

Special cases: critical points (fixed points).

$$u_* = \text{const}, \qquad F(u_*) = 0$$

Let $u(t) = u_*(t) + \eta(t)$

$$u' = u_*' + \eta' = F(u_* + \eta)$$

$$\underbrace{u_*'}_{F(u_*)} + \eta' = F(u_*) + \sum_{j=1}^{n} \frac{\partial F}{\partial u_j}(u_*)\, \eta_j + O(\eta^2)$$

$$\eta_i' = \sum_{j=1}^{n} \frac{\partial F_i}{\partial u_j}(u_*)\, \eta_j + O(\eta^2)$$

Define: $A_{ij}(t) = \dfrac{\partial F_i}{\partial u_j}(u_*(t))$ Linearization matrix about $u_*(t)$

Solution to $\eta' = A\eta$ (neglect η^2)
is

$$\eta = \exp\left(\int_0^t A(\tau)\,d\tau\right)\eta(0)$$

Assume fixed point ($u_* =$ const., so A const.)

matrix exponential

$$e^B = I + B + \tfrac{1}{2}B^2 + \dots$$

If A is diagonalisable, $A = S\Lambda S^{-1}$:

$$\exp(At) = S e^{\Lambda t} S^{-1},$$ so

$$(S^{-1}\eta) = e^{\Lambda t}(S^{-1}\eta(0))$$

Eigenvalues $\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ determine stability.

$\mathrm{Re}\,\lambda_i < 0,\ \forall i \Rightarrow$ asymptotically stable

$\mathrm{Re}\,\lambda_i \leq 0,\ \forall i$, some $\mathrm{Re}\,\lambda = 0 \Rightarrow$ neutrally stable (Hamiltonian systems)

$\mathrm{Re}\,\lambda_j > 0$, for any $j \Rightarrow$ unstable

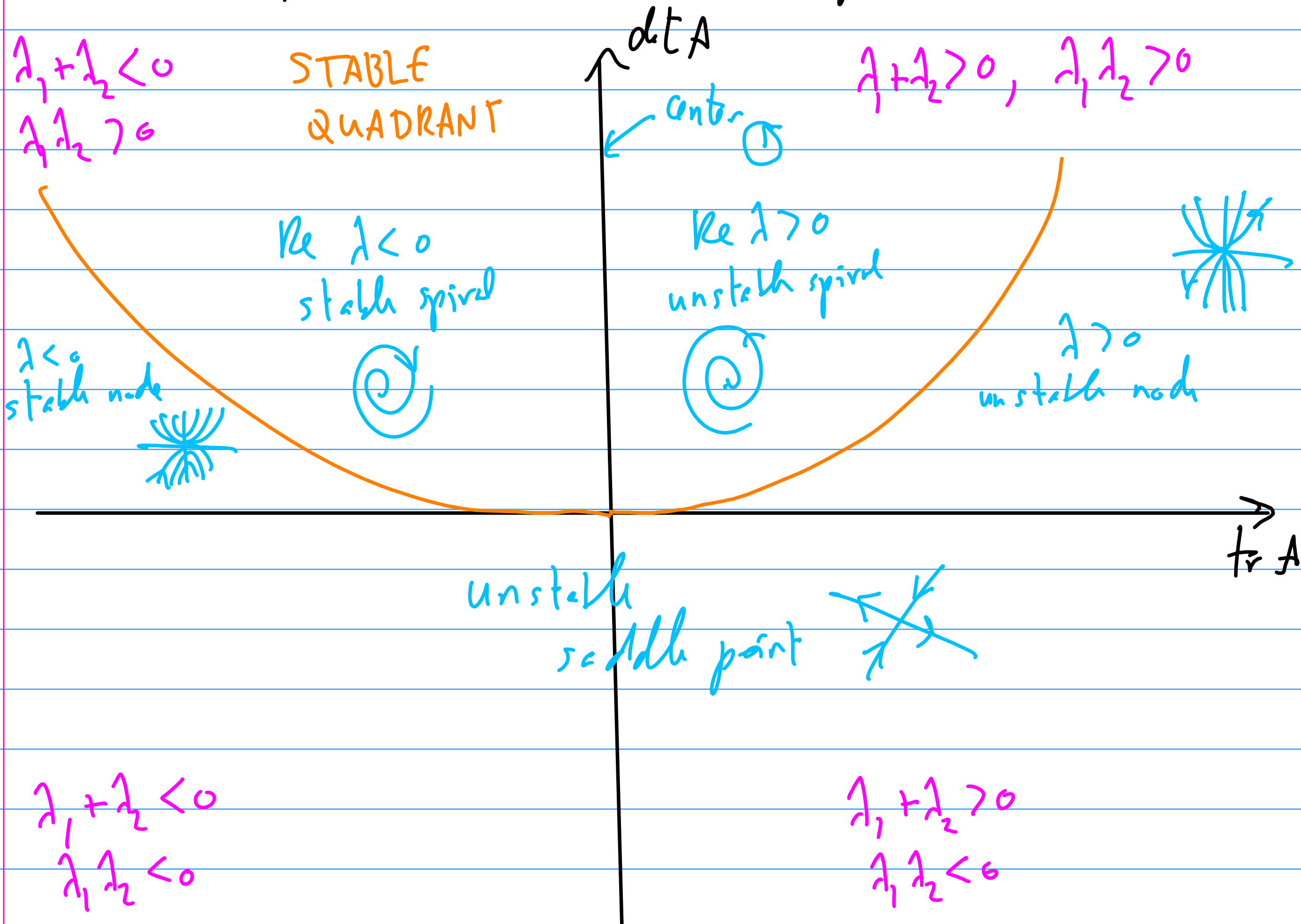
Take $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (2×2)

$$\det(A - \lambda I) = (a-\lambda)(d-\lambda) - bc$$
$$= \lambda^2 - (a+d)\lambda + (ad-bc)$$
$$= \lambda^2 - (\operatorname{tr} A)\lambda + (\det A)$$

characteristic polynomial

So eigenvalues are $\lambda_\pm = \dfrac{\operatorname{tr} A \pm \sqrt{(\operatorname{tr} A)^2 - 4(\det A)}}{2}$

Stability requires negative trace and positive determinant.



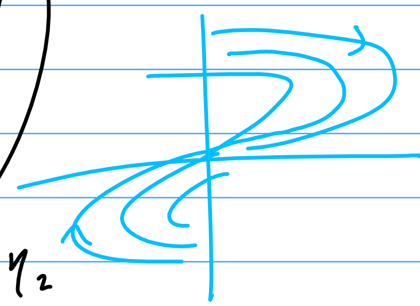
Note that the trace and determinant are not enough to characterize the "marginal" cases (orange curve).

Marginal cases: $(\mathrm{tr}\,A)^2 = 4\det A$, so

$$\lambda_\pm = \frac{1}{2}\,\mathrm{tr}\,A = \lambda \qquad \text{degenerate}$$

Diagonalizing A gives either

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$



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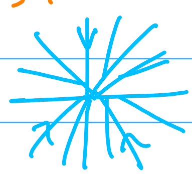

$\lambda < 0$

$$\dot\eta_1 = \lambda\eta_1 + \eta_2$$
$$\dot\eta_2 = \lambda\eta_2$$

$$\hookrightarrow \eta_1 = te^{\lambda t}, \quad \eta_2 = e^{\lambda t}$$

Fun fact: solving $\dot\eta = A\eta \Rightarrow \eta(t) = e^{At}\eta(0)$

$$p(\lambda) = \lambda^2 - (\mathrm{tr}\,B)\lambda + \det B \qquad B = At$$

Important special case: $\mathrm{tr}\,A = 0$ $(\Rightarrow \mathrm{tr}\,B = 0)$

Cayley-Hamilton theorem: B satisfies $p(B) = 0$.

So $B^2 = -(\det B)\,I$

Let $\Delta = \det B$.

$$e^B = \sum_{n=0}^{\infty} \frac{B^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{B^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{B^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-\Delta)^n}{(2n)!} I + \sum_{n=0}^{\infty} \frac{(-\Delta)^n B}{(2n+1)!}$$

$$= I \sum_{n=0}^{\infty} \frac{(\sqrt{-\Delta})^{2n}}{(2n)!} + \frac{B}{\sqrt{-\Delta}} \sum_{n=0}^{\infty} \frac{(\sqrt{-\Delta})^{2n+1}}{(2n+1)!}$$

$$= \cosh\sqrt{-\Delta}\, I + \frac{B}{\sqrt{-\Delta}} \sinh\sqrt{-\Delta}$$

$$\Delta = \det B = t^2 \det A$$

unstable for $\det A < 0$

$$e^{At} = \cosh\left(\sqrt{-\det A}\, t\right) I + \frac{A}{\sqrt{-\det A}} \sinh\left(\sqrt{-\det A}\, t\right)$$

For $\det A > 0$, $\cosh\left(\sqrt{-\det A}\, t\right) = \cos\left(\sqrt{\det A}\, t\right)$

$$\frac{\sinh\left(\sqrt{-\det A}\, t\right)}{\sqrt{-\det A}} = \frac{\sin\left(\sqrt{\det A}\, t\right)}{\sqrt{\det A}}$$

This expression applies even when A is not diagonalizable.