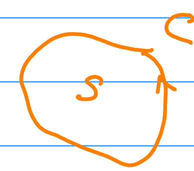
Lecture 15: Complex methods (end)

Some final thoughts on complex variable methods:

- For point vortex/source, can interpret singularity at the origin as a δ function of circulation:

$$\nabla^2 u = M\,\delta(\underline{x}) \quad \text{or} \quad \nabla^2 s = \Gamma\,\delta(\underline{x})$$

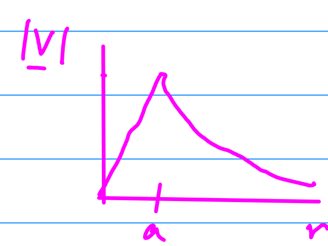


since

$$\oint_C \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = \int_S \nabla^2 u\,da = \int_S M\,\delta(\underline{x})\,da = M$$

and similarly for point vortex.

- Physically, point vortex can be thought of as arising from some finite motion:



$$\underline{V}(x,y) = \begin{cases} \dfrac{\Gamma}{2\pi}\dfrac{\hat{\theta}}{r^2}, & r > a \\[2ex] \dfrac{\Gamma}{2\pi a^3}\, r\hat{\theta}, & r \lesssim a \end{cases}$$

continuous at $r = a$

solid body motion

This is the Rankine vortex. $a \to 0$ recovers point vortex.

- For magnetic systems Γ is always zero: there are no point sources of magnetic fields (no magnetic monopoles). (At least we've never found one!)

- Cauchy's integral formula:

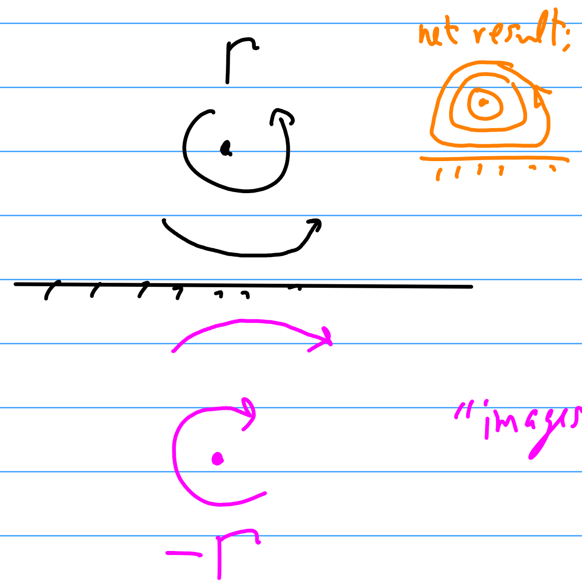
$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\,dz \quad , \; f(z) \text{ analytic in } C, \; a \text{ inside } C$$

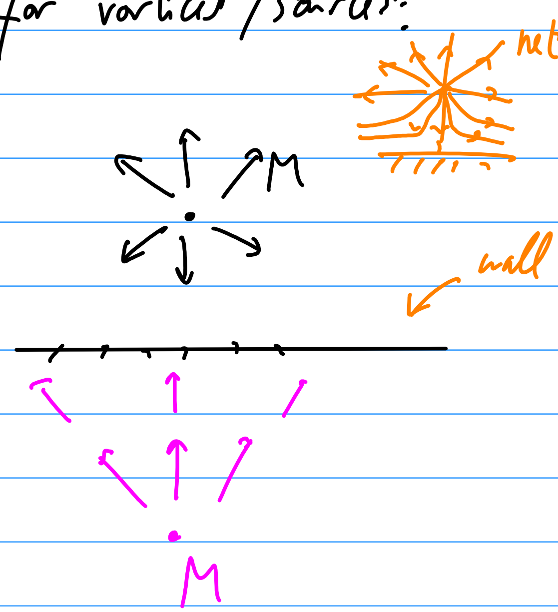
leads to a maximum principle:

The maximum of $|f(z)|$ in S is reached on C.

[The hottest point is on the boundary. See Strang p. 355 for short proof.]

- Image methods work great for vortices/sources:





These satisfy condition $\underline{V}\cdot\hat{y} = 0$ at $y=0$

- General functions (not analytic):

Take $f(x,y)$ any complex function. $(x, y \in \mathbb{R})$

$z = x + iy$, $\bar{z} = x - iy$ $\Rightarrow$ $x = \frac{1}{2}(z+\bar{z})$, $y = \frac{1}{2i}(y - \bar{y})$

So $f(x,y) \to F(z,\bar{z})$

$$\nabla^2 = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$ ← show this!

If F is analytic, then $F = F(z)$ (or $\bar{z}$) only, and

$$\frac{\partial^2}{\partial z \partial \bar{z}} F = 0$$

Stokes flow

Bi-harmonic: $(\nabla^2)^2 f = 0 \Rightarrow 16 \frac{\partial^4 F}{\partial z^2 \partial \bar{z}^2} = 0$

Can solve analytically by integration:

$$\frac{\partial^3 F}{\partial z^2 \partial \bar{z}} = g_1''(z), \quad \frac{\partial^2 F}{\partial z \partial \bar{z}} = g_1'(z) + g_2'(\bar{z})$$

$$\frac{\partial F(z,\bar{z})}{\partial \bar{z}} = g_1(z) + z g_2'(\bar{z}) + g_3'(\bar{z})$$

$$F(z,\bar{z}) = \bar{z} g_1(z) + z g_2(\bar{z}) + g_3(\bar{z}) + g_4(z)$$

Here $g_{1,2,3,4}$ are arbitrary analytic functions.