Lecture 14: Complex variables (cont'd)

In fluid mechanics, complex variable methods chiefly apply to potential flow in 2D:

$$\underline{V} = \begin{pmatrix} U(x,y) \\ V(x,y) \end{pmatrix} \quad \text{with} \quad \begin{aligned} \nabla \cdot \underline{V} &= 0 \\ \nabla \wedge \underline{V} &= 0 \end{aligned}$$

This means $\dfrac{\partial U}{\partial x} + \dfrac{\partial V}{\partial y} = 0$, $\quad \dfrac{\partial U}{\partial y} - \dfrac{\partial V}{\partial x} = 0$

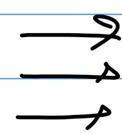
Can "build in" these relations by defining

$$\underline{V} = \begin{pmatrix} \partial u/\partial x \\ \partial u/\partial y \end{pmatrix} = \begin{pmatrix} \partial v/\partial y \\ -\partial v/\partial x \end{pmatrix}$$

Cauchy-Riemann equation!

So any analytic function gives a velocity field.

$f(z) = z$ is the "constant" flow,

since $u = x$, $v = y$, so $\underline{V} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ 

In general $f(z) = e^{-i\alpha} z$ is a flow at angle α.

u is called the potential, s the stream function.

$f(z) = u + is$ is the complex potential.

Note that $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial s}{\partial x} = U - iV$

components of velocity

So $\boxed{U + iV = \overline{f'(z)}}$

Now consider:

$$f(z) = \frac{\Gamma}{2\pi i} \log z \qquad \Gamma \text{ constant}$$

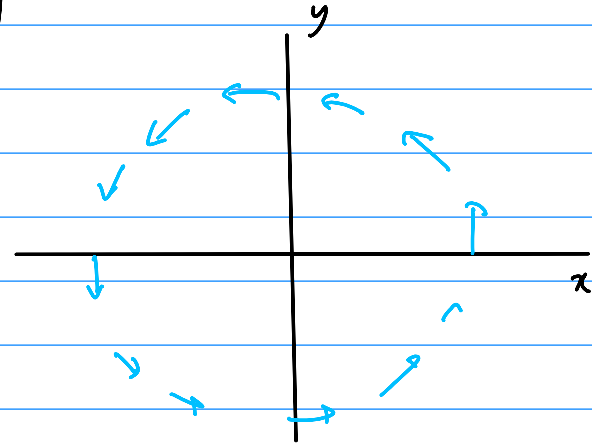
Then $U - iV = f'(z) = \dfrac{\Gamma}{2\pi i z}$

$$= \frac{\Gamma}{2\pi |z|^2}(-i\bar{z}) = \frac{\Gamma}{2\pi r^2}(-y - ix)$$

$r = |z|$

So: $\begin{pmatrix} U \\ V \end{pmatrix} = \dfrac{\Gamma}{2\pi r^2}\begin{pmatrix} -y \\ x \end{pmatrix}$

This is the velocity field for a point vortex or vortex filament (3D)



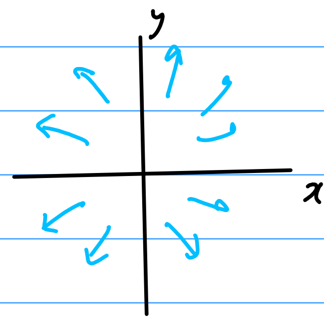
In magnetism, this would correspond to a magnetic monopole, which does not (we think) exist.

Why? Because at the origin this $f(z)$ is not analytic. This is ok for fluid dynamics since potential flow is an approximation but not so for electromagnetism, where $\nabla \cdot \underline{B} = 0$ is a law of nature.

$$f(z) = \frac{M}{2\pi} \log z \quad \text{has} \quad \begin{pmatrix} U \\ V \end{pmatrix} = \frac{M}{2\pi r^2} \begin{pmatrix} x \\ y \end{pmatrix}$$

This is a point source

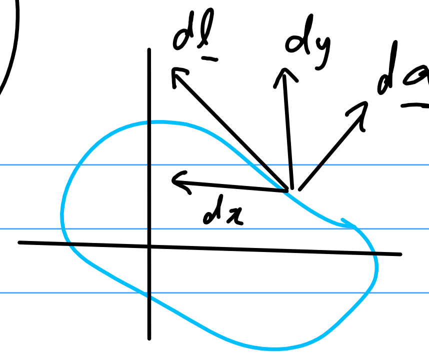
In electromagnetism: point charge.



Note that in principle: $\displaystyle\int_C f'(z)\, dz = 0$ when $f'(z)$ is analytic.

However, here it is not analytic at the origin!

$$d\underline{l} = \begin{pmatrix} dx \\ dy \end{pmatrix}, \quad d\underline{a} = \begin{pmatrix} dy \\ -dx \end{pmatrix}$$



$$\int_C \underline{V} \cdot d\underline{l} = \int_C U\,dx + V\,dy$$

$$= \int_C \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = \int_C du$$

$$\int_C \underline{V} \cdot d\underline{a} = \int_C U\,dy - V\,dx$$

$$= \int_C \frac{\partial s}{\partial y}\,dy + \frac{\partial s}{\partial x}\,dx = \int_C ds$$

Hence,

$$\int_C df = \int_C du + i \int_C ds$$

$$= \Gamma + i M$$

$$\boxed{\int_C f'(z)\,dz = \Gamma + iM}$$

M = mass flux

Γ = circulation

C is any contour enclosing origin

For point vortex,

$$\int_C f'(z)\,dz = \int_C \frac{\Gamma}{2\pi i z}\,dz$$

$$= 2\pi i\,[\text{residue at } z=0]$$

$$= \Gamma \,!$$

(no mass flux)

For point source,

$$\int_C f'(z)\,dz = \int_C \frac{M}{2\pi z}\,dz$$

$$= 2\pi i\left(\frac{M}{2\pi}\right) = iM$$