Lecture 12: Analytical methods (cont'd)

Another example: start from heat equation:

$$\frac{\partial u}{\partial t} = \nabla^2 u \qquad \text{(before } u(x,y,t) \to u(x,y)\text{)}$$

Let $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$:

$$-u_t + u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

$$-T'R\Theta + T\Theta\left(R'' + \frac{R'}{r}\right) + \frac{RT\Theta''}{r^2} = 0$$

$$\frac{T'}{T} = \frac{1}{R}\left(R'' + \frac{R'}{r}\right) + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \text{const} = -\lambda$$

So $T \sim e^{-\lambda t}$

Now for an initial value problem we have $\lambda \geq 0$, so solutions don't blow up.

$$\frac{r^2}{R}\left(R'' + \frac{R'}{r}\right) + \lambda r^2 = -\frac{\Theta''}{\Theta} = m^2$$

to get periodic solutions in θ

So $\Theta \sim \sin m\theta, \cos m\theta$

Finally,

$$r^2 R'' + rR' + (\lambda r^2 - m^2) R = 0$$

If we let $\rho = \sqrt{\lambda}\, r$, $B(\rho) = R(\rho/\sqrt{\lambda})$:

$$\rho^2 B'' + \rho B' + (\rho^2 - m^2) B = 0$$

This is Bessel's equation

This is solved by series solution (for instance).

Solutions are $J_m(\rho)$, $Y_m(\rho)$

blows up at 0, so don't use here

To apply a boundary condition at $r=1$, say $u(1,\theta,t)=0$, need to impose

$$J_m(\sqrt{\lambda}) = 0$$

which means $\sqrt{\lambda}$ is a zero of J_m.

$$\sqrt{\lambda} = \rho_{m,n} \leftarrow n^{th} \text{ zero of } J_n$$

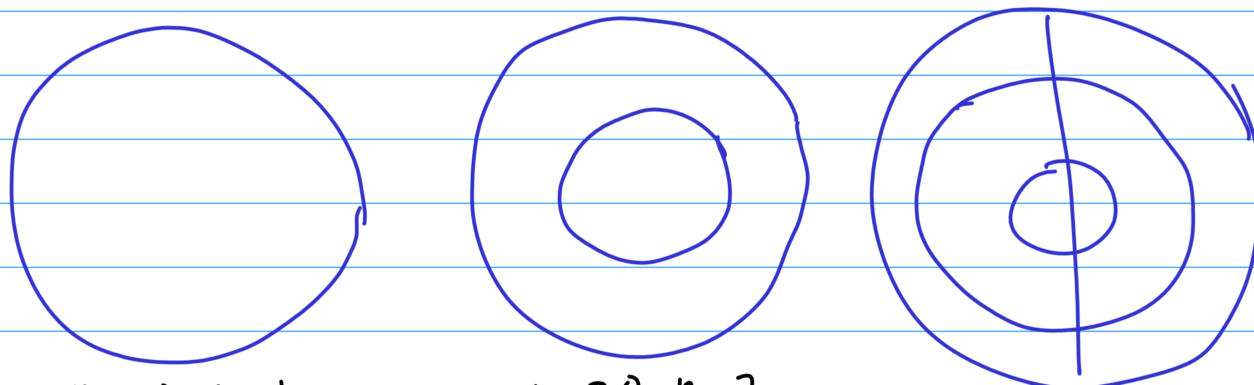
These values are tabulated. For a drum ($u_{tt} = \nabla^2 u$), they give the eigenfrequencies of sound.

Note that a vibrating drum problem ($u_{tt} = \nabla^2 u$) has a similar separated solution:

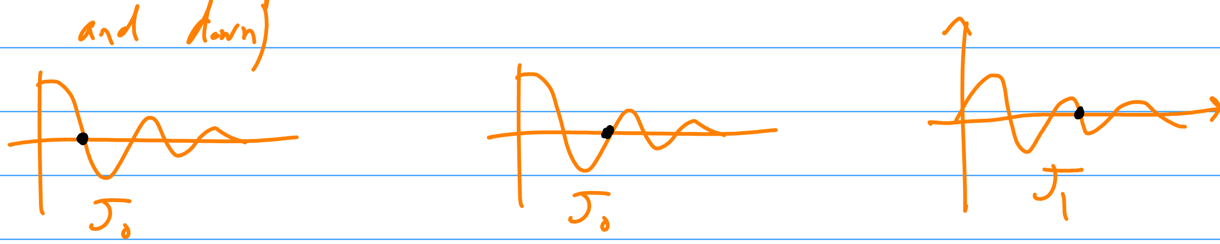
$$T'' + \lambda T = 0$$

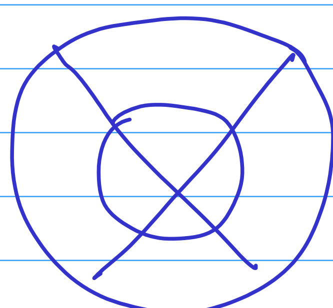
except now $T \sim \sin\sqrt{\lambda}\, t,\ \cos\sqrt{\lambda}\, t.$

Useful to visualize using nodal lines ($u = 0,\ \forall t$).



$m=0,\ n=1$ (drum moves up and down) $\qquad m=0,\ n=2 \qquad m=1,\ n=3$





$m=2,\ n=2$

m, n give the number of zeros of u in θ and r.

For the heat equation, nodal lines are lines of zero temperature.

Fourier transforms:

Fourier series $\left(f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \right)$ occur when x take on values over a finite domain: $x \in [a,b]$, $a,b<\infty$, or $f(x)$ periodic.

For infinite and semi-infinite domains, use Fourier transforms:

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk$$

Fourier pair (definitions differ)

There is a correspondence between inner products in x and k:

$$2\pi \int_{-\infty}^{\infty} f(x) \bar{g}(x) dx = \int_{-\infty}^{\infty} \hat{f}(k) \bar{\hat{g}}(k) dk$$

This gives Plancherel's formula:

$$2\pi \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk$$

(width, standard deviation)

We can estimate the "spread" of a function by treating

$$p(x) = \frac{|f(x)|^2}{\int_{-\infty}^{\infty} |f(x)|^2 dx}, \quad P(k) = \frac{|\hat{f}(k)|^2}{\int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk}$$

as probability densities. The variance is then,

$$W_x^2 = \int_{-\infty}^{\infty} (x^2 - \langle x \rangle^2)\, p(x)\, dx, \quad \langle x \rangle = \int_{-\infty}^{\infty} x\, p(x)\, dx$$

$$W_k^2 = \int_{-\infty}^{\infty} (k^2 - \langle k \rangle^2)\, P(k)\, dk, \quad \langle k \rangle = \int_{-\infty}^{\infty} k\, P(k)\, dk$$

Then we have: $\boxed{W_x W_k \geq \frac{1}{2}}$, for any f (with finite W_x)

proof: Translate x, k so that $\langle x \rangle = \langle k \rangle = 0$, which does not affect the width.

$$\underbrace{\left| \int x \overline{f(x)}\, f'(x)\, dx \right|^2} \leq \underbrace{\left(\int |xf|^2 dx \right) \left(\int |f'|^2 dx \right)}_{\text{Cauchy-Schwartz inequality}}$$

$$= \left| \underbrace{\left[x \frac{|f|^2}{2} \right]_{-\infty}^{\infty}}_{0, \text{ if integrals exist}} - \int \frac{|f|^2}{2} dx \right|$$

$$\int |f'|^2 dx = \frac{1}{2\pi}\int |k\hat{f}|^2 dk$$

$$\left(\int |xf|^2 dx\right)\left(\int |k\hat{f}|^2 dk\right) \geq \frac{1}{4}\, 2\pi \left|\int f^2 dx\right|^2$$

$$= \frac{1}{4}\left(\int |f|^2 dx\right) \times \left(2\pi \int |\hat{f}|^2 dk\right)$$

Hence, $W_x^2 W_k^2 \geq 1/4$.

This means that functions that are "narrow" in x are "wide" in k.

Best example: $f(x) = e^{-x^2/2\sigma^2} \Rightarrow \hat{f}(k) = \sqrt{2\pi}\,\sigma\, e^{-\frac{1}{2}\sigma^2 k^2}$
$(\sigma > 0)$

Gaussian of width σ $(W_x = \sigma/\sqrt{2})$

Gaussian of width $1/\sigma$ $(W_y = 1/\sqrt{2}\sigma)$

For a Gaussian, $W_x W_k = 1/2$.

Interesting exercise: show that a Gaussian is the only such function!

$\Rightarrow$ minimize $P(f) = W_x W_k$

This ties into Heisenberg uncertainty principle.

In quantum mechanics,

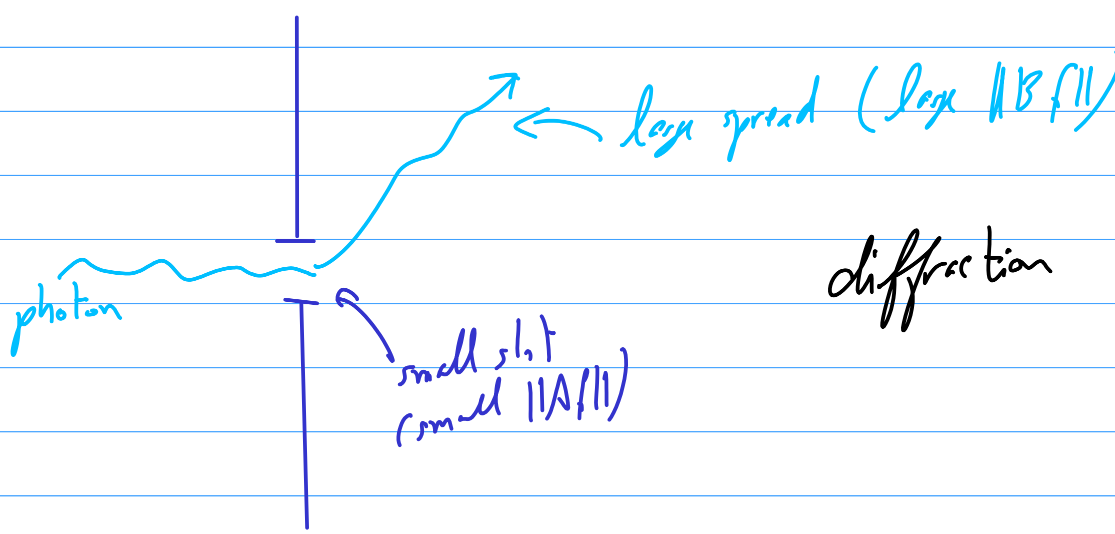
$$Af = xf \qquad \text{"position operator"}$$

$$Bf = f' \qquad \text{"momentum operator"}$$

$$\|Af\|^2 = \int |Af|^2 \, dx \, , \quad \|Bf\|^2 = \int |Bf|^2 \, dx$$

Can show: $\|Af\| \, \|Bf\| \geq \frac{1}{2} \left| f^T \underbrace{[A,B]}_{\substack{\text{commutator} \\ AB - BA}} f \right|$

$$= \frac{1}{2} \quad \text{since } [A,B] = -I$$



diffraction