Lecture 11: Principle of least action; Analytical methods

In dynamics least energy is replaced by least action

For elastic bar,

$$A(u) = \int_{t_0}^{t_1} (K - P)\, dt$$

kinetic, potential

$$= \int_{t_0}^{t_1} dt \int_0^1 dx \left(\frac{1}{2}\rho \left(\frac{du}{dt}\right)^2 - \frac{1}{2} c \left(\frac{du}{dx}\right)^2 \right)$$

density $\rho(x)$

Can use the same technique as before: $u \to u + v$, then enforce $A(u)$ being a minimum. Get condition

$$\frac{\partial}{\partial t}\left(\rho \frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial x}\left(c \frac{\partial u}{\partial x}\right)$$

(with appropriate B.C.)

For $\rho = c =$ const., this is $u_{tt} = \frac{c}{\rho} u_{xx}$, → wave speed $V^2 = \frac{c}{\rho}$

the wave equation

$A^T C A u = f$

So here: $A = \begin{pmatrix} -\partial/\partial t \\ \partial/\partial x \end{pmatrix}$, $A^T = \begin{pmatrix} \partial/\partial t & -\partial/\partial x \end{pmatrix}$, $C = \begin{pmatrix} \rho & \\ & c \end{pmatrix}$

Analytical methods:

example: solve $\nabla^2 u = 0$, $(x,y) \in$ unit disk
$u(1,\theta) = u_0(\theta)$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Separation of variables: $u = R(r)\Theta(\theta)$

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

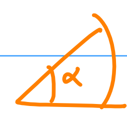
$$\Theta R'' + \Theta \frac{R'}{r} + \frac{R}{r^2}\Theta'' = 0$$

$$\frac{r^2}{R}\left(R'' + \frac{R'}{r}\right) = -\frac{\Theta''}{\Theta} = \text{const} = \lambda^2$$

Θ equation: $\Theta'' + \lambda^2 \Theta = 0$

$$\Theta = \sin \lambda\theta, \; \cos \lambda\theta$$

Since θ periodic, $\lambda = m \in \mathbb{Z}$

[Different for wedge problem:  .]

R equation: $R \sim r^p$: $R'' + \frac{1}{r}R' - \frac{R}{r^2}\lambda^2 = 0$

$$p(p-1) + p - \lambda^2 = 0 \iff p^2 = \lambda^2, \; p = \pm\lambda$$

Hence, r solutions are $R \sim r^{\lambda}, r^{-\lambda}$

require boundedness at $r=0$, so do not include for $\lambda > 0$.

When $\lambda = 0$ there is only one solution: $R \sim$ const.

The other solution: $\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0 \Rightarrow rR' = c_1$

$\Rightarrow R = c_1 \log r + c_2$

but again blows up at $r=0$

So finally:

$$u(r,\theta) = \sum_{m=0}^{\infty} r^m \left(a_m \cos m\theta + b_m \sin m\theta\right)$$

$$= \sum_{m=-\infty}^{\infty} c_m r^{|m|} e^{im\theta}$$

complex

$$u(1,\theta) = \sum_{m=-\infty}^{\infty} c_m e^{im\theta} = u_0(\theta)$$

$$\sum_{m=-\infty}^{\infty} c_m \underbrace{\int_0^{2\pi} e^{im\theta} e^{-in\theta}\, d\theta}_{2\pi\delta_{m,n}} = \int_0^{2\pi} u_0(\theta) e^{-in\theta}\, d\theta$$

$2\pi\delta_{m,n} \rightarrow$ orthogonal functions

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} u_0(\theta) e^{-in\theta}\, d\theta$$

This is a series solution.

Can also write in terms of integral solution:

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} u_0(\varphi)\,\frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)}\,d\varphi$$

Why? Because

$$\frac{1}{2}\,\frac{1-r^2}{1+r^2-2r\cos\theta} = \frac{1}{2} + r\cos\theta + r^2\cos 2\theta + \cdots$$

So after changing $\cos k(\theta-\varphi) = \cos k\theta\cos k\varphi + \sin k\theta\sin k\varphi$, we see each term is a coefficient of the Fourier series.

So, for example, if $u_0(\varphi) = \delta(\varphi)$,

$$u(r,\theta) = \frac{1}{2\pi}\,\frac{1-r^2}{1+r^2-2r\cos\theta}$$

$u(r,\theta)$ can be regarded as steady temperature distribution due to a point source of heat, or electric field, ...

Note that the orthogonal functions $e^{in\theta}$ arose naturally from the equations.