Lecture 9: Continuous case (cont'd)

Recall: $P(x) = \frac{1}{2}(Ax)^T C (Ax) - x^T f$

This was minimized at equilibrium.

Continuous analogue: $\underset{\text{functional}}{P(u)} = \int_0^1 \left(\frac{c}{2}\left(\frac{du}{dx}\right)^2 - f(x)u \right) dx$

with $u(0) = a$.

Let's find the min. Consider $P(u+v)$, where $(u+v)(0) = a$, so $v(0) = 0$.

$$P(u+v) = \int_0^1 \left(\frac{c}{2}\left(\frac{d(u+v)}{dx}\right)^2 - f(x)(u+v) \right) dx$$

$$= P(u) + \int_0^1 \left(c\,\frac{du}{dx}\frac{dv}{dx} + \frac{c}{2}\left(\frac{dv}{dx}\right)^2 - f(x)v \right) dx$$

If u is a minimum of $P(u)$, require

$$P(u+v) - P(u) \geq 0, \quad \forall v.$$

Hence, $\int_0^1 \left[c\,\frac{du}{dx}\frac{dv}{dx} - fv \right] dx + \underbrace{\int_0^1 \frac{c}{2}\left(\frac{dv}{dx}\right)^2 dx}_{\geq 0}$

$\geq 0, \quad \forall v$

If the term linear in v is $-$ve, can always make quadratic term negligible; also, $+$ve always gives a negative counterpart ($v \to -v$), so really must vanish identically:

$$(*) \quad \int_0^1 \left[c \frac{du}{dx} \frac{dv}{dx} - f v \right] dx = 0 \quad \forall \text{ test } v$$

Now integrate by parts:

$\delta P / \delta u$

$$\int_0^1 \left[-\frac{d}{dx}\left(c \frac{du}{dx} \right) - f \right] v \, dx$$
$$+ \left[c \frac{du}{dx} v \right]_0^1 = 0 \quad (**)$$

Choose $v(x) = \delta(x - x')$,

this is our equilibrium eq.

$$\Rightarrow \quad -\frac{d}{dx}\left(c \frac{du}{dx} \right) = f \quad , \quad \forall x$$

This is the Euler-Lagrange equation for the minimization problem. Also the strong form of $(*)$, which is the weak form.

Still need boundary term in $(**)$ to vanish: $v(0) = 0$ ok,

$$\Rightarrow \quad c(1) \frac{du}{dx}(1) = 0 \qquad \text{natural B.C.}$$

• Note that jumps in $c(x)$ are handled by breaking up the integral in the weak form, then doing integration by parts. We are led to a jump condition:

$$c_- u_-' = c_+ u_+' \quad \text{at the jump.}$$

$$[cu'] = 0$$

• If $f(x)$ has a concentrated force at x_0,

$$f(x) = \delta(x - x_0), \quad \text{then}$$

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} -\frac{d}{dx}\left(c\frac{du}{dx}\right)dx = \int_{x_0-\varepsilon}^{x_0+\varepsilon} f(x)\,dx = 1$$

$$-\left(c_+ u_+' - c_- u_-'\right) = 1 \quad \text{jump condition}$$

Often written $[cu'] = 1$

• There is also a complementary energy

$$Q(w) = \int_0^1 \frac{1}{2c} w^2\,dx, \quad -\frac{dw}{dx} = f(x)$$

$$w(1) = 0$$

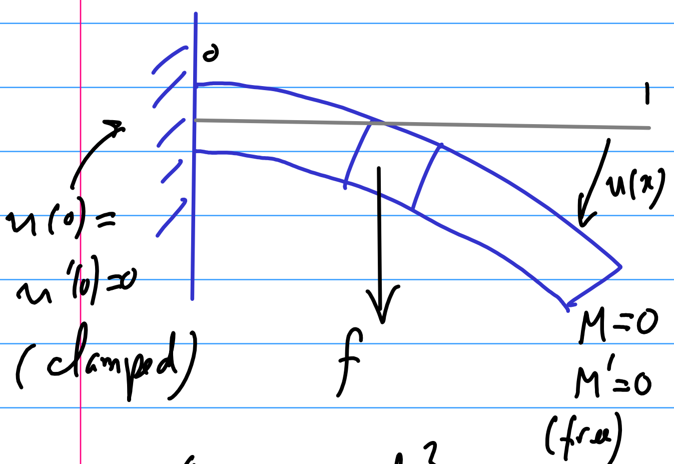
We can do constrained minimization:

$$L(u,w) = \int_0^1 \left[\frac{w^2}{2c} + u\left(\frac{dw}{dx} + f\right)\right] dx$$

Lagrange multipliers $u(x)$

We find $\min_{w} Q(w) = \max_{u} \left(-P(u)\right)$ duality

The structure $A^T C A u = f$ applies in many other cases. For instance, a beam can bend but not stretch:



displacement $u(x)$

curvature $u''(x)$ why? $\dfrac{u''}{(1+u'^2)^{3/2}}$

moment $c u'' = M$

$$A u = \frac{d^2 u}{dx^2}$$

$$A^T C A u = f \Rightarrow \frac{d^2}{dx^2}\left(c \frac{d^2 u}{dx^2}\right) = f$$

no "−" sign why?

torque balance

So for $f = c = 1$,

$$u'''' = 1 \Rightarrow u = \frac{1}{4!}x^4 + c_3 x^3 + c_2 x^2 + \cancel{c_1}^{\,0} x + \cancel{c_0}^{\,0}$$

$$M = u'' = \frac{1}{2}x^2 + 6c_3 x + c_2 = 0 \quad \text{at } x = 1$$

$$M' = u''' = x + 6c_3 = 0 \quad \text{at } x = 1 \qquad c_3 = -\frac{1}{6}$$

$$\frac{1}{2} - 1 + c_2 = 0 \Rightarrow c_2 = 1/2$$

$$u(x) = \frac{1}{24}x^4 - \frac{1}{6}x^3 + \frac{1}{2}x^2$$

max deflection is

$$u(1) = \frac{1}{24} - \frac{1}{6} + \frac{1}{2} = \frac{3}{8}$$

Can also put PDEs in A^TCA form.

$$A = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} \qquad \begin{pmatrix} -\partial/\partial x & -\partial/\partial y \end{pmatrix} \begin{pmatrix} c_1 & \\ & c_2 \end{pmatrix} \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} u = f$$

For $c_1 = c_2 = \text{const}$ this is Poisson's equation

$$-\nabla^2 u = f$$

or Δ

For $f = 0$ it is Laplace's equation.

These arise in a simplified version of fluid dynamics.

In 2D we write $v = \begin{pmatrix} \partial u/\partial x \\ \partial u/\partial y \end{pmatrix} = \nabla u$ $\quad$ v is the fluid velocity at (x,y)

This is for irrotational flow, since $\nabla \wedge v = \nabla \wedge \nabla u = 0$

Then $\nabla \cdot v = 0 \Rightarrow \nabla^2 u = 0$

incompressibility condition