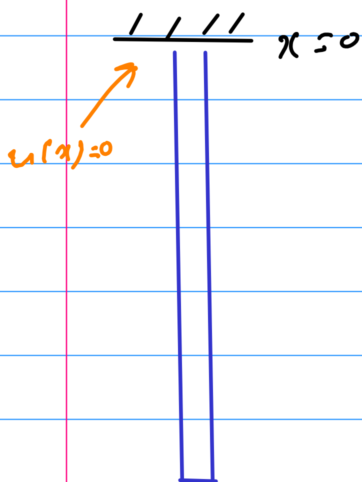
Lecture 8: The continuous case

Notation changes slightly:

$u(x)$ is displacement of a small element from equilibrium.



i.e., element at x is displaced to $x + u(x)$.

strain $e = \dfrac{du}{dx}$ [careful: e was elongation]

force in bar is $w = c\dfrac{du}{dx}$

equilibrium requires $\left(c\dfrac{du}{dx}\right)_{x+\Delta x} - \left(c\dfrac{du}{dx}\right)_{x} + f\Delta x = 0$

force per unit length

As $\Delta x \to 0$, this is

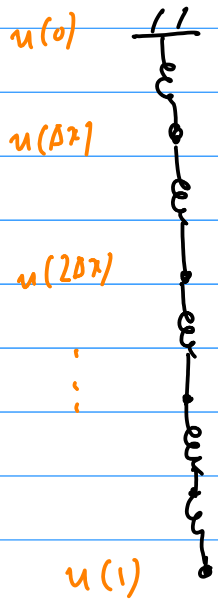
$$-\frac{d}{dx}\left(c\,\frac{du}{dx}\right) = f$$

This is the equivalent to $A^T C A x = f$.

$$A = \frac{d}{dx}, \qquad A^T = -\frac{d}{dx}$$

Why? Think of a chain of masses:

central difference



$$e = Au = \frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x}$$

$$A = \frac{1}{2\Delta x}\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & 1 & \\ & -1 & 0 & \\ & & & \ddots \end{pmatrix}$$

$u(0)=0$

$$A^T = \frac{1}{2\Delta x}\begin{pmatrix} 0 & -1 & \\ 1 & & -1 \\ & 1 & \ddots \end{pmatrix} = -A$$

There is a boundary condition $u(0) = 0$.

At the bottom $(x=1)$ there is no force: the last spring is unstretched because the mass hanging from it is infinitesimal:

$$w(1) = c(1)\frac{du}{dx}(1) = 0.$$

force-free boundary cond. (Neumann)

Think more carefully about transpose: for matrices, define A^T by

$$x^T A y = (A^T x)^T y \quad \forall x, y.$$

For operators like $\frac{d}{dx}$, need inner product:

$$(e, w) \;\; (= e^T w) \;\; = \int_0^1 e(x) w(x)\, dx$$

Then define A^* (or A^T), the <u>adjoint operator</u>, by

$$(Ae, w) = (e, A^* w)$$

For $A = \frac{d}{dx}$,

$$(Au, w) = \int_0^1 \frac{du}{dx} w \, dx$$

$$= -\int_0^1 u \frac{dw}{dx} dx + \overbrace{\left[uw\right]_{x=0}^{1}}^{\text{boundary term}}$$

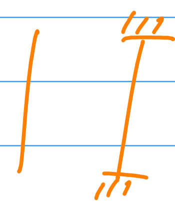
$$= (u, A^* w)$$

The boundary term vanishes here, since $u(0)=0$, $w(1)=0$.

Hence, since the relationship holds for all e, w: "weakly"

$$A^* w = -\frac{dw}{dx} \quad \text{for} \quad u(0) = w(1) = 0.$$

or other combinations such that $[uw]_0^1 = 0$.



example: uniform bar with $f=1$, $c=1$, $u(0)=w(1)=0$:

$$-\frac{d}{dx}\left(c\frac{du}{dx}\right)=f \Rightarrow w = c\frac{du}{dx} = \int_x^1 f\,dx$$

$$= 1-x$$

$$u = \frac{1}{c}\int_0^x w\,dx = x - \frac{1}{2}x^2$$

So the displacement of the bottom point is $u(1) = 1/2$.
The total length of the bar is $1 + 1/2 = 3/2$.

In this case we could solve for w immediately: this is the "square A" (statically determinate) case.

$\Rightarrow$ $A = \frac{d}{dx}$ with one B.C. is invertible.

Can invert each part of A^TCA

(boundary condition)

If we fix both ends, then $u(0)$ and $u(1)$ are specified ($m>n$, statically indeterminate)

A has 2 B.C., A^T none.

Take $u(0)=0$, $u(1)=b$ for the problem above.

$$cu'' = -f \quad (u''=-1) \Rightarrow u = -\frac{1}{2}x^2 + c_1 x + c_2$$

$$= \frac{1}{2}x(1-x) + bx$$

Now find w: $w = c\frac{du}{dx} = b + \frac{1}{2} - x$

$b=0$: top half in tension ($w>0$), bottom half in compression ($w<0$)

$w(0) = b + \frac{1}{2}$, $w(1) = b - \frac{1}{2}$

So in this last case we cannot solve for w without finding u first.

Also, note that $[uw]_0^1 = -b(b-\tfrac{1}{2}) \neq 0$ in this last case.

$$\left[[uw]_0^1 = 0 - u(1)w(1) = 0 - b(b - \tfrac{1}{2}) \right]$$

Hence $A^T \neq -A$ and we shouldn't have been able to use "$A^T C A x = f$". However, this was the pre-stressed case ($e = Ax + b$) so actually

$$A^T C (Ax + b) = f \iff A^T C A x + A^T C b = f$$

In the continuous notation: $A^T C A u + A^T C(bx) = f$

(I think)

If we define

$$(Au, w) = \int_0^1 \frac{du}{dx} w \, dx + u(0)w(0)$$

then $$(u, A^* w) = \int_0^1 u \frac{dw}{dx} dx + u(1)w(1)$$

for any boundary conditions.

Always have $(Au, w) = (u, A^* w)$

More generally, we will encounter

$$-\frac{d}{dx}\left(c\frac{du}{dx}\right) + qu = f$$

Same form as before $(A^T C A)$, except now

$$A = \begin{pmatrix} \frac{d}{dx} & 1 \end{pmatrix}^T, \quad A^* = \begin{pmatrix} -\frac{d}{dx} & 1 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{d}{dx} & 1 \end{pmatrix} \begin{pmatrix} c(x) & 0 \\ 0 & q(x) \end{pmatrix} \begin{pmatrix} d/dx \\ 1 \end{pmatrix} = -\frac{d}{dx}\left(c\frac{d}{dx}\right) + q$$

Assume B.C. satisfy $\left[cu\frac{du}{dx} \right]_0^1 = 0$.

This is a Sturm-Liouville problem.