Lecture 7: Trusses (cont'd)

Trusses: N nodes, r constraints, m bars

$$n = 2N - r \text{ degrees of freedom}$$

Last time:

$$e = A_0 x \qquad (A_0 \text{ is } m \text{ by } 2N)$$

when a row of A_0 contains $\pm\cos\theta, \pm\sin\theta$.

[In 3D, $\pm\cos\theta_1$, $\pm\cos\theta_2$, $\pm\cos\theta_3$, direction cosines]

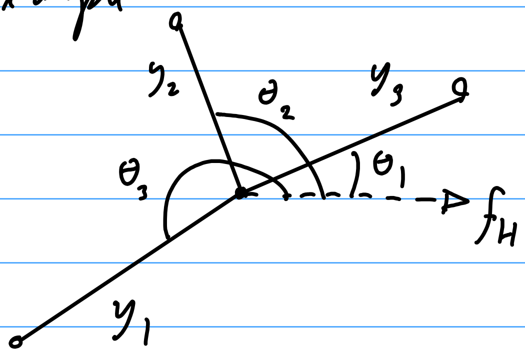
Note the sum of each row is still 0.

Now drop the r columns corresponding to each fixed displacement:

Get A, m by n.

$A^T y = f$ will then give the force balance.

For example



$$-y_1\cos\theta_1 - y_2\cos\theta_2 - y_3\cos\theta_3 = f_H$$

(etc., + one for f_V vertical)

$y_1 = y_2 = 0$, $y_3 > 0$, $\theta_3 \in [-\pi/2, \pi/2]$

$\Rightarrow$ $f_H < 0$ left for tension

Finally, we have $y = Ce$, where C contains the material properties of each bar.

$$K = A^T C A \quad \text{stiffness matrix.}$$

Solving $f = Kx$ gives x in terms of f
$x \to x_1 \dots x_n$, $f \to f_1 \cdots f_n$

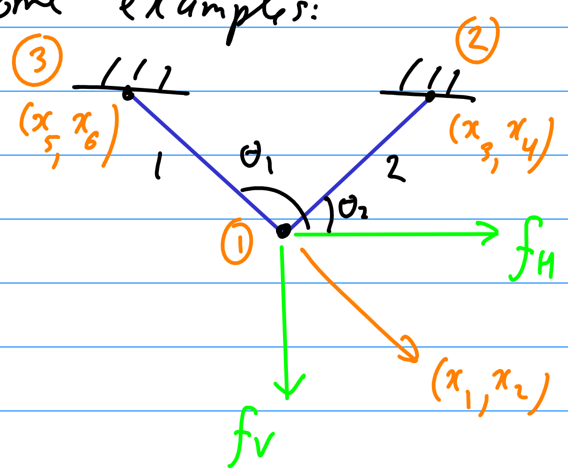
Then find $y_1 \cdots y_m$ using $A^T y = f$.

Go back to $A_0^T y = f$: n of these are satisfied.

The remaining r equations can be solved for $f_{m+1} \cdots f_{2N}$, given $y_1 \cdots y_m$, which gives the force at the supports.

Some examples:

1)



$m = 2$, $N = 3$, $r = 4$
$\Rightarrow n = 2N - r = 2$

Define down as positive

$$e_1 = \left(\overset{0}{x_5}\cos\theta_1 - x_1\cos\theta_1 - \overset{0}{x_6}\sin\theta_1 + x_2\sin\theta_1\right) + O(x^2/L)$$

$$e_2 = \left(\overset{0}{x_3}\cos\theta_2 - x_1\cos\theta_2 - \overset{0}{x_4}\sin\theta_2 + x_2\sin\theta_2\right) + O(x^2/L)$$

$$e = Ax = \begin{pmatrix} -\cos\theta_1 & \sin\theta_1 \\ \cos\theta_2 & \sin\theta_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

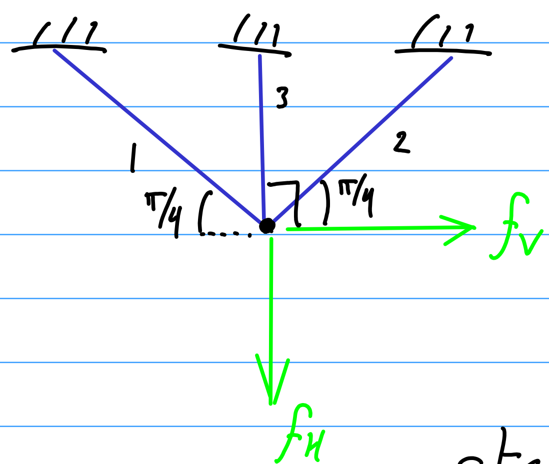
Now note that $A^T y = f$ alone is enough to find the forces. determinate

$\begin{pmatrix} f_H \\ f_V \end{pmatrix}$

For $\theta_2 = \pi/4$, $\theta_1 = 3\pi/4$, $\Rightarrow$ $A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

$$K = A^T C A = \frac{1}{2}\begin{pmatrix} c_1 + c_2 & c_1 - c_2 \\ c_1 - c_2 & c_1 + c_2 \end{pmatrix}$$

2)



Now $m = 3$, so A is 3 by 2.

$A^T y = f$

$$\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

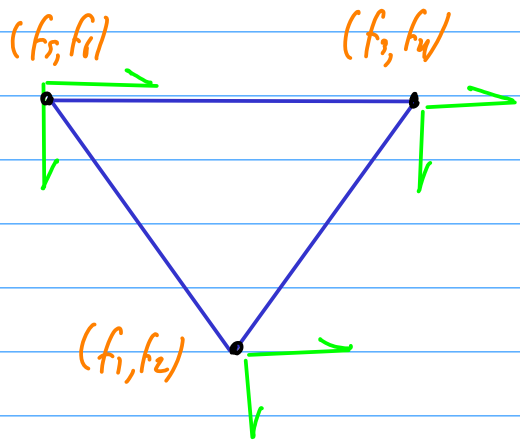
statically indeterminate

Cannot solve $A^Ty = f$ directly for y.
Must first solve $Kx = f$.

$$K = A^TCA = \frac{1}{2}\begin{pmatrix} c_1+c_2 & c_1-c_2 \\ c_1-c_2 & c_1+c_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & c_3 \end{pmatrix}$$

extra vertical bar

3)



$m = 3$, $N = 3$, $r = 0$

$n = 2N - r = 6$

A is 3 by 6.

$\Rightarrow$ overdetermined (typically no solution)

Bars undergo rigid motion $\rightarrow$ no static equilibrium

$$A^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & \sqrt{2} \\ 0 & -1 & 0 \\ -1 & 0 & -\sqrt{2} \\ -1 & 0 & 0 \end{pmatrix}$$

When do we have a solution to $A^Ty = f$? When f is in the range of A^T.

$R(A^T) \perp N(A)$

$N(A)$ spanned by $(1,0,1,0,1,0)$, $(0,1,0,1,0,1)$, $(1,-1,0,-2,0)$
Hence:

$f_1 + f_3 + f_5 = 0$, $f_2 + f_4 + f_6 = 0$, $f_1 - f_2 - 2f_4 = 0$

H force balance — V force balance — torque balance (prevent rotation)

These are the conditions for the existence of a static solution.

More on torque balance (force moment):

$$\sum_i \underline{r}^{(i)} \wedge \underline{f}^{(i)} = (0,0)\wedge(f_5,f_6) + (\sqrt{2}L, 0)\wedge(f_3,f_4) + \left(\tfrac{\sqrt{2}}{2}L, \tfrac{\sqrt{2}}{2}L\right)\wedge(f_1,f_2)$$

position from, say, node 3.

$$= \sqrt{2}L f_4 + \frac{\sqrt{2}}{2}L(f_2 - f_1)$$

$$= -\frac{\sqrt{2}}{2}L\left(f_1 - f_2 - 2f_4\right)$$

$$= 0, \text{ for static equilibrium.}$$

Summary:

The columns of A are linearly independent (stable truss)

(1) Statically determinate $(m=n)$

(2) " indeterminate $(m>n)$

The columns of A are linearly dependent (unstable truss)

(3) Rigid motion (too few supports)

(4) Mechanism (the truss deforms) → see Strang

In continuum mechanics we would use strain ε and stress σ:

$$\varepsilon = \frac{e}{L} \quad \text{[dimensionless]} \qquad \sigma = \frac{y}{A} \quad \text{[force per unit area]}$$

cross-sectional area of bar

Then $\sigma = E\varepsilon$ is Hooke's law,

where E depends only on the material (not on shape)

E is called Young's modulus (units of stress)

The elastic constant is then $c = \dfrac{y}{e} = \dfrac{\sigma A}{\varepsilon L} = \dfrac{EA}{L}$,

which depends on the geometry of each bar (A, L).

Recall $e = Ax + b$. $b \neq 0 \Rightarrow$ pre-stressed

bar is stretched before inserted in truss.