Lecture 6: Duality and trusses

nonlinear constraint!

Classic example: minimize $Q = y^T K y$ subject to $y^T y = 1$.

$$L = Q + x(y^T y - 1)$$

$$\frac{\partial L}{\partial y} = 2(Ky + xy) = 0 \Rightarrow Ky = -xy$$

So x is an eigenvalue of K.

Since K symmetric, eigenvalues are real, so to minimize take smallest eigenvalue, $x = -\lambda$, with e-vec y:

Then $\min L = y^T K y = \lambda y^T y = \lambda$.

$\max L$ gives the matrix 2-norm of K.

Equivalently: min/maximize ratio $y^T K y / y^T y$.

Duality. Primal problem:

Minimize $Q(y) = \frac{1}{2} y^T C^{-1} y - b^T y$ subject to $A^T y = f$.

$$\longrightarrow \quad L = Q + x^T (A^T y - f)$$

Lag. mult. vector

$$\frac{\partial L}{\partial y} = C^{-1}y - b + Ax = 0$$

$$\Rightarrow \quad y = C(b - Ax)$$

$$\min L(x,y) = -\frac{1}{2}(b-Ax)^T C(b-Ax) - x^T f$$

$$=: \underline{\underline{-P(x)}}$$ $P(x)$ is an "energy"

Dual problem:

Maximize $-P(x) = -\frac{1}{2}(Ax-b)^T C(Ax-b) - x^T f$

(no constraint!)

Now we show that Dual = Primal.

First, for any x and y such that $A^T y = f$, we have

$$Q(y) \geqslant -P(x).$$ "weak duality" $Q(y) \geqslant L(x,y) \geqslant -P(x)$ with "Frisch"

Why? Because $-P$ is the minimum of L!
So $L(x,y) \geqslant -P(x)$, and $L = Q$ at the min.

Now take $Q + P$: (use $A^T y = f$)

$$Q+P = \frac{1}{2}y^T C^{-1} y - b^T y + \frac{1}{2}(Ax-b)^T C(Ax-b) + x^T f$$

$$= \frac{1}{2}(C^{-1}y + Ax - b)^T C \underbrace{(C^{-1}y + Ax - b)}_{z}$$

$$\geqslant 0 \, !$$

Hence, $Q + P = \frac{1}{2} z^T C z$ has a min. at $z=0$.

$$\Rightarrow C^{-1} y + Ax - b = 0.$$

We reach "full duality" when $z=0$ $(Q = -P)$

Summary: constrained min. of $Q(y)$ subject to $A^T y = f$ equals the max of $-P(x)$. The minimizing y and maximizing x (saddle pt. of L, where $Q+P=0$) satisfy:

$$C^{-1} y + Ax = b \quad , \quad A^T y = f$$

Return to spring example: (y = internal forces)

$$Q = \frac{1}{2} \frac{y_1^2}{c_1} + \frac{1}{2} \frac{y_2^2}{c_2} + \frac{1}{2} \frac{y_3^2}{c_3} = \frac{1}{2} y^T C^{-1} y$$

energy in springs

with constraint (force balance)

$$y_1 - y_2 = f_1 \ , \quad y_2 - y_3 = f_2 \qquad (A^T y = f)$$

Minimize Q subject to constraint:

$$L = Q - x_1 (y_1 - y_2 - f_1) - x_2 (y_2 - y_3 - f_2)$$

Minimizing gives $y = CAx \Rightarrow A^T C A x = f$

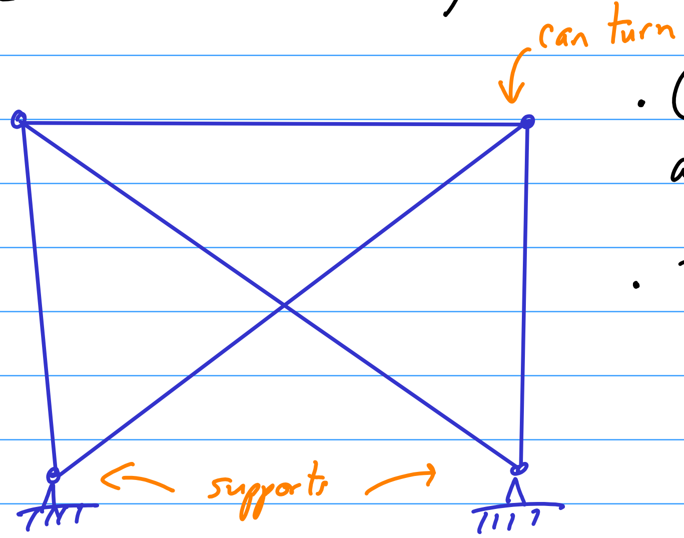
Could also minimize $P = \frac{1}{2} x^T A^T C A x - x^T f$.

total potential energy

Structures in equilibrium:

trusses: system of idealized bars that doesn't bend.

Consider 2D configurations.



- Give external forces f_j at each node
- Two displacements x_j for each node.

m bars with N nodes. $2N$ nodal displacements (2 directions at each node)

Each support fixes 2 displacements, total of r.

So remains $n = 2N - r$ degrees of freedom.

$x_1, \dots, x_n$ ⟵ includes horizontal and vertical disp

Similarly, there are $2N$ components of forces at the nodes.

We do not know the forces at the supports, but the other $2N - r$ forces are prescribed.

$$f_1, \dots, f_n \qquad \text{applied forces}$$

Forces $y_1, \dots, y_m$ in the bars.

$y_i > 0$ is in tension

No direction needed in the graph

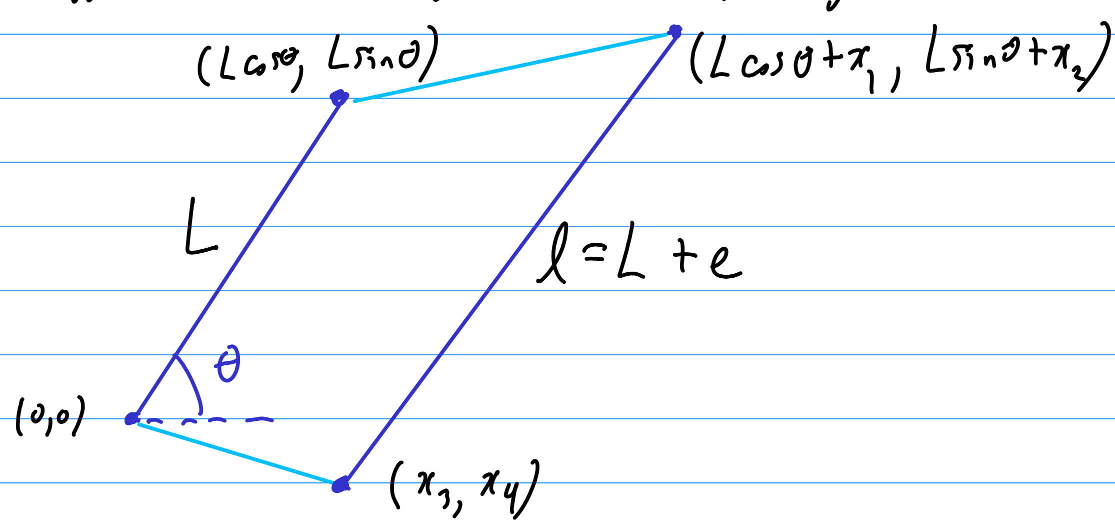
$y_i < 0$ is in compression

Recall displacements $e = \underbrace{b}_{\text{usually } 0} - Ax$.

Convention in mechanics is $e = Ax$.
So a positive force is in the direction of positive displacement.

A_0 has m rows, one for every bar.

Suppose the ends of a bar of original length L move.



$$l^2 = (L\cos\theta + x_1 - x_3)^2 + (L\sin\theta + x_2 - x_4)^2$$

To leading order in x:

$$l^2 = L^2 + 2L(x_1\cos\theta - x_3\cos\theta + x_2\sin\theta - x_4\sin\theta) + O(x^2)$$

$$l = L + (\quad " \quad) + O(x^2/L)$$

$e = l - L$ = elongation

We thus have a typical term in $A_0 x$:

Nonzero entries of each row are $\pm\cos\theta$, $\pm\sin\theta$,

[In 3D, $\pm\cos\theta_1$, $\pm\cos\theta_2$, $\pm\cos\theta_3$, direction cosines]

Note the sum of each row is still 0.

Now drop the r columns corresponding to each fixed displacement:

Get A, m by n.

$A^T y = f$ will then give the force balance.