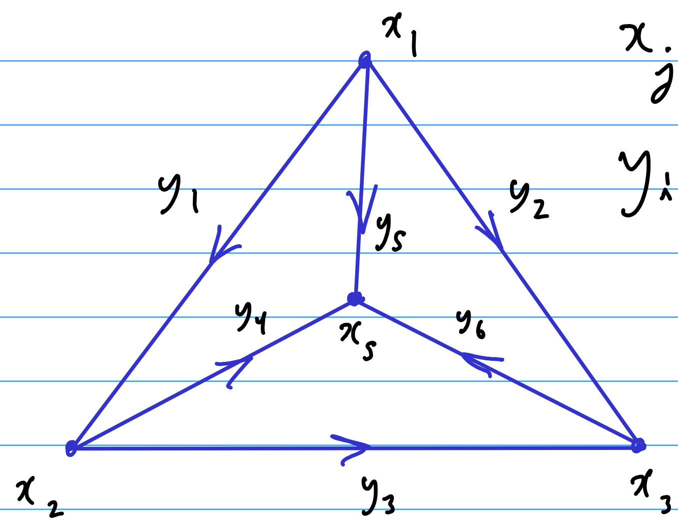
Lecture 5: Fundamental equations for equilibrium

The framework discussed last time for graphs is useful in many physical systems:



x_j: potential at node j (N)

y_i: flow on edge i (m)

The y_i are determined by the potential differences across the edges and by physical properties of edges.

Physical properties usually represented by $m \times m$ matrix C, often $\mathrm{diag}(c_1, \ldots, c_m)$.

In any case C is symmetric +ve def.

y connected to x by A_0 $(m \times N)$.

A_0 gives the geometry of the network

Use $+1/-1$ edge convention from last lecture.

For network above,

edge-node incidence matrix (connectivity m., topology m.)

$$A_0 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \text{ edges}$$

nodes

The colums of A_0 are not linearly independent: cannot get unique solution to $Ax=b$.

This is the arbitrary potential at each node discussed in last lecture.

To eliminate this ambiguity, ground one node: $x_N = 0$.

Then (drop last column in A_0)

$$A = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

is m by $n = N-1$

$Ax=b$ has unique solution.

For mechanical networks n is the total number of degrees of freedom at the nodes, not counting any nodes that are fixed.

(In other words, eliminate fixed nodes, for which x is known.)

The vectors b and f cause things to happen.

b gives voltage sources.

Then: $e = b - Ax$ will lead to flow.

Ohm's law or Hooke's law is then

$$y = Ce = C(b - Ax)$$

$b = 0$ for Hooke's law

Finally, the second fundamental equation is

$$A^T y = f$$

Kirchoff's current law ($f = 0$)

Note that springs have $b = 0$, while circuits have $f = 0$.

These can be grouped into the fundamental equations for equilibrium:

$$\begin{pmatrix} C^{-1} & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} b \\ f \end{pmatrix}$$

or

$$\underbrace{\begin{pmatrix} C^{-1} & A \\ 0 & -A^T C A \end{pmatrix}}_{\text{indefinite}} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} b \\ f - A^T C b \end{pmatrix}$$

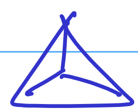
So we get m positive pivots and n negative pivots.

For a network,

$$(A^TCA)_{jk} = -c_i \quad \text{if edge } i \text{ connects } j \neq k$$

$$(A^TCA)_{kk} = \sum c_i \quad \text{over edges meeting node } k.$$

For our earlier example: $A^TCA = \begin{pmatrix} c_1+c_2+c_3 & -c_1 & -c_2 \\ -c_1 & c_1+c_3+c_4 & -c_3 \\ -c_2 & -c_3 & c_2+c_3+c_6 \end{pmatrix}$



Lagrange multipliers:

example: minimize $Q = \frac{1}{2}(y_1^2 + y_2^2)$, subject to the

constraint $2y_1 - y_2 = 5$.

Lagrange multiplier

Solve this by defining $L = Q + x_1 (2y_1 - y_2 - 5)$

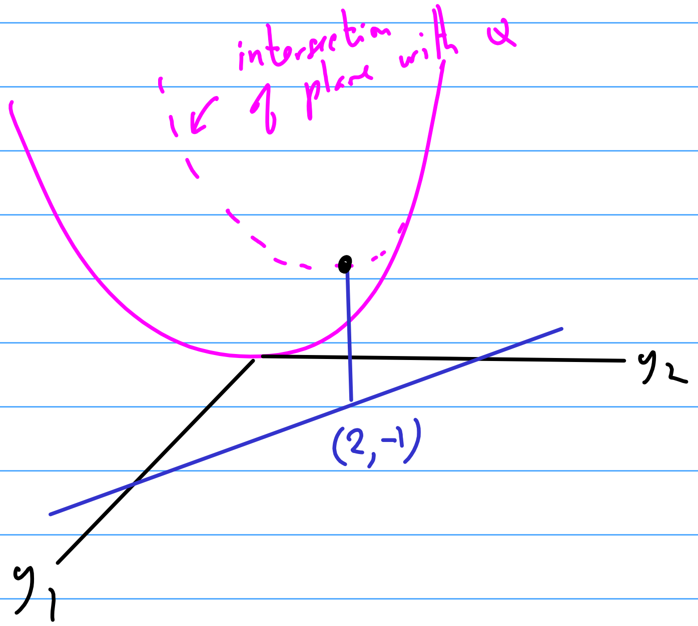
(Note $L = Q$ when constraint is satisfied)

$\partial L / \partial y_1 = y_1 + 2x_1 = 0$ minimize $L(y_1, y_2, x_1)$

$\partial L / \partial y_2 = y_2 - x_1 = 0$

$\partial L / \partial x_1 = 0 + (2y_1 - y_2 - 5) = 0$ constraint satisfied!

Solution is $(y_1, y_2, x_1) = (2, -1, -1)$.



Now minimize

$$Q = \frac{1}{2} y^T C^{-1} y - b^T y$$

with constraint $A^T y = f$.

Here A is m by n, so m constraints.

Need Lagrange multipliers $x_1, \dots, x_m$

$$L = Q + x^T \underbrace{(A^T y - b)}_{\text{constraint}}$$

$$\frac{\partial L}{\partial y_i} = \frac{\partial Q}{\partial y_i} + \frac{\partial}{\partial y_i}(x^T A^T y) = 0$$

$$x^T A^T y = \sum x_j A_{kj} y_k$$

$$\frac{\partial(\uparrow)}{\partial y_i} = \sum_j x_j A_{ij} = (Ax)_i$$

$$\Rightarrow \quad C^{-1} y - b + Ax = 0$$

Together with the constraint itself, this is

$$\begin{pmatrix} C^{-1} & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} b \\ f \end{pmatrix}$$

These are the fundamental equations for equilibrium!