Lecture 4: Incidence matrices of graphs

Recall some facts about $Ax=b$: $\quad A \quad m \times n$

$$(Ax)_i = \underset{\substack{\uparrow \\ \text{column 1 of } A}}{A_{i1}} x_1 + \cdots + A_{in} x_n$$

Ax is a combination of the columns of A.

Ax, $x \in \mathbb{R}^n$ forms the column space $R(A)$.

$Ax=b$ has a solution iff $b \in R(A)$.

The set $Ax=0$, $x \in \mathbb{R}^n$ is the nullspace $N(A)$.

$Ax=b$ has a unique solution (if it has one at all) if $N(A)=0$.

In general, two solutions $Ax_1=b$, $Ax_2=b$ differ by an element of the nullspace.

$$\left(Ax_1 - x_2 = 0 = A(x_1 - x_2) \text{ so } (x_1 - x_2) \in N(A) \right)$$

The fundamental subspaces of A are

$$R(A),\ R(A^T),\ N(A),\ N(A^T)$$

Fundamental theorem of linear algebra:

$$\dim R(A^T) + \dim N(A) = r + (n-r) = n$$

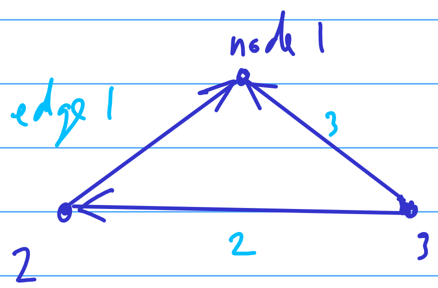
rank

$$\dim R(A) + \dim N(A^T) = r + (m-r) = m$$

The spaces $R(A^T)$ and $N(A)$ are orthogonal

" " $R(A)$ and $N(A^T)$ " "

<u>An example</u>: edge-node incidence matrix of a graph



$$A = \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}}_{\text{nodes}} \Bigg\} \text{edges}$$

To solve $Ax = b$:

$$x_1 - x_2 = b_1$$
$$x_2 - x_3 = b_2$$
$$x_1 - x_3 = b_3$$

Add up first two: $(x_1 - x_2) + (x_2 - x_3) = x_1 - x_3$

This implies: $b_1 + b_2 = b_3$ for a solution to exist.

This is the requirement $b \in R(A)$.

If the nodes are potentials in a circuit, then this is Kirchoff's voltage law.

There is a nullspace generated by $(1,1,1)$, so the solution to $Ax=b$ is not unique.

Physically: cannot determine potentials from potential differences.

Now consider $A^T y = f$: A^T is n by m (here 3×3):

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} y_1 + y_3 \\ y_2 - y_1 \\ -y_2 - y_3 \end{pmatrix}$$

$N(A^T)$ generated by $(1,1,-1)$.

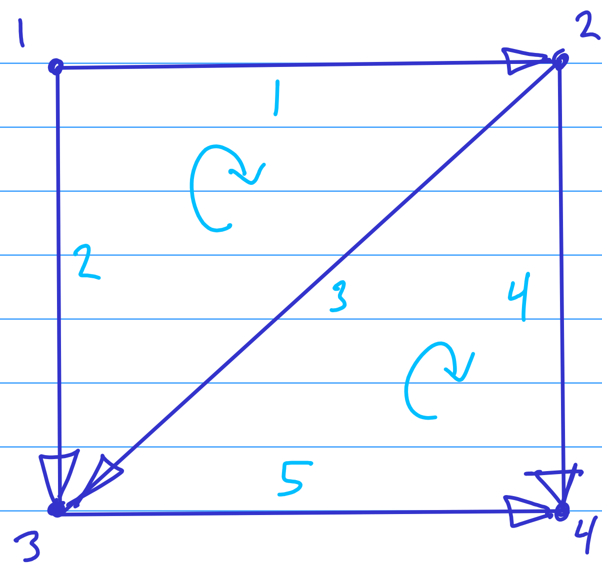
Need f in the range $R(A^T)$, or $\perp$ to $N((A^T)^T) = (1,1,1)$ as found previously

$\Rightarrow f_1 + f_2 + f_3 = 0.$

Here $y_1 + y_3$ is the flow into node 1
$y_2 - y_1$ " " " " " 2
$-y_2 - y_3$ " " " " " 3

sum is zero, no net flow.

A more complicated graph:



$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$N(A) = c(1,1,1,1) \qquad c \in \mathbb{R}$$

Still get 0 total voltage change around the nodes

But now $N(A^T)$ is given by
$$\begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = 0$$

Solutions are generated by
$$y_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad y_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

These correspond to the two cycles in the graph!

Indeed, for $Ax = b$ to have a solution we must have $b \in R(A)$, or equivalently $b \perp N(A^T)$.

Hence, $y_1^T b = y_2^T b = 0$

$\Rightarrow \quad b_1 - b_2 + b_3 = 0 \quad , \quad -b_3 + b_4 - b_5 = 0.$

These are Kirchoff's law for the two loops!

$Ax = b$ voltage law, can be solved when $y_1^T b = y_2^T b = 0$

$A^T y = f$ current law, " " " " $(1,1,1,1)^T f = 0$

$(f_1 + f_2 + f_3 + f_4 = 0)$

In general, for a connected graph with m edges and n nodes, incidence matrix has rank $r = n-1$.

The null vector is always $(1, 1, \ldots, 1)$.

$N(A^T)$ has dimension $m - r = m - n + 1$ loops of the graph (Euler's formula)

Summary of matrix factorizations:

1. $A = LDU$ (with symmetry: $U = L^T$)

$$\left.\begin{array}{ll} 2. & PA = LDU \\ 3. & PA = LU \end{array}\right\} \text{with row exchange}$$

echelon form

4. $A = S\Lambda S^{-1}$ diagonalization of A (A square, diagonalizable)

   If $A^TA = AA^T$, $S \to Q$ (orthonormal)

   NORMAL MATRIX

   A Hermitian $\Rightarrow$ Λ real

5. $A = MJM^{-1}$ Jordan form (any square A)

6. $A = QR$ Q orthogonal, R triangular (A has indep. columns)

   Gram-Schmidt orthogonalization

7. $A = Q_1 \Sigma Q_2^T$ Singular value decomposition (any A)

8. $A = QB$ B symmetric polar decomposition (A invertible)