Lecture 3: Eigenvalues and dynamical systems

Recall: $Ax = \lambda x$ $\qquad$ Solve for λ in $\qquad$ characteristic polynomial

$$\det(A - \lambda I) = 0.$$

Typically, find n linearly independent eigenvectors

$$A \underbrace{\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}}_{S} = \begin{pmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{pmatrix}$$

$$AS = S\Lambda \qquad \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$\boxed{S^{-1}AS = \Lambda \quad \text{or} \quad A = S\Lambda S^{-1}}$$

S is invertible since columns are lin. indep.

Distinct eigenvalues $\Rightarrow$ lin. indep. eigenvectors

proof: Let $x_k = \sum_{j=1}^{l} a_j x_j$, with x_j lin. indep.

and $Ax_k = \lambda_k x_k$, $Ax_j = \lambda_j x_j$, $j = 1 \dots l$

$$Ax_k = \lambda_k x_k \iff \sum_j a_j \lambda_j x_j = \sum_j \lambda_k a_j x_j$$

or $\sum_j a_j(\lambda_j - \lambda_k)\, x_j = 0$.

Violates assumption of linear indp. unless λ_k = at least one of the λ_j.

For Hermitian matrices ($\bar{A}^T = A$), the eigenvalues are real and eigenvectors orthogonal.

Can then write: $A = Q \Lambda Q^T$ (for $A = A^T$)

where $Q^{-1} = Q^T$ (Q orthogonal)

Compare to $A = LDL^T$ ← pivots

The pivots are not the same as eigenvalues, but +ve for +ve-def.

proof: $Ax = \lambda x \Rightarrow \bar{x}^T A x = \bar{x}^T x \lambda$

transpose: $\bar{x}^T \underbrace{\bar{A}^T}_{A} = \bar{x}^T \bar{\lambda} \Rightarrow \bar{x}^T A x = \bar{x}^T x \bar{\lambda}$

$\Big\}\ \lambda = \bar{\lambda}$.

Now for eigenvectors: let $Ax_1 = \lambda_1 x_1$ and $Ax_2 = \lambda_2 x_2$

$\bar{x}_2^T A x_1 = \lambda_1 \bar{x}_2^T x_1$ $\qquad$ $\bar{x}_2^T \underbrace{\bar{A}^T}_{A} x_1 = \lambda_2 \bar{x}_2^T x_1$

$\Rightarrow (\lambda_1 - \lambda_2)\, \bar{x}_2^T x_1 = 0$

So if $\lambda_1 \neq \lambda_2$, $\bar{x}_2^T x_1 = 0 \Rightarrow$ orthogonal

(If $\lambda_1 = \lambda_2$ consider a small perturbation)

Note that an anti-Hermitian ($\bar{A}^T = -A$) matrix (or skew-symmetric or antisymmetric in the real case) has

$$\lambda = -\bar{\lambda} \implies \lambda \text{ pure imaginary (or 0).}$$

For a real matrix, they must come in pairs since

$$Ax = \lambda x \implies \bar{A}\bar{x} = \bar{\lambda}\bar{x} \qquad \lambda, \bar{\lambda} \text{ both eigenvalues.}$$

Eigenvectors of different pairs are orthogonal, since in proof above for skew-symmetric get $(\lambda_1 + \lambda_2)\bar{x}_2^T x_1 = 0$.

Recall springs from last time: $\quad f - Kx = 0$

Now do not assume static.

assume same mass for now (see end of lecture)

Then $\quad F = Ma \implies ma = f - Kx$

a: d^2x/dt^2 or $\ddot{x}$

$$\boxed{\ddot{x} + Ax = b} \qquad A = K/m, \quad b = f/m$$

Now we have $A = Q\Lambda Q^T$, so let $y = Q^T x$:

$$\implies \ddot{y} + \Lambda y = Q^T b$$

diagonalized

$$\ddot{y}_k + \lambda_k y_k = (Q^T b)_k$$

$$y_k(t) = a_k \cos(\sqrt{\lambda_k}\, t) + b_k \sin(\sqrt{\lambda_k}\, t) + \text{(particular solution)}$$

Then transform back to x using $x = Qy$.

The solutions to $\ddot{y}_k + \lambda_k y_k = 0$ are called <u>normal modes</u>

(If A is not +ve def then get exponentials or linear in time)

The constants a_k, b_k are set by initial condns.

The spring system from last time had

$$K = \begin{pmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3+c_4 \end{pmatrix}$$

Let $c_1 = c_2 = c_3 = k$

Then $$A = \underbrace{\frac{k}{m}}_{\omega^2} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Eigenvalues are $\lambda_1 = 2 - \sqrt{2}$, $\lambda_2 = 2$, $\lambda_3 = 2 + \sqrt{2}$.

Eigenvectors. $x_1 = (1, \sqrt{2}, 1)$, $x_2 = (1, 0, -1)$, $x_3 = (1, -\sqrt{2}, 1)$.

- In mode 1 they all move in phase, but middle mass has larger amplitude.
- In mode 3 the middle mass is out of phase
- In mode 2 the middle mass doesn't move.

Note that we need very special initial conditions for the system to be in one of these exact states. Usually we will get a "mixture".

More generally, consider

$$\frac{d^2u}{dt^2} + p\frac{du}{dt} + qu = 0$$

Trick: let $v = \frac{du}{dt}$, so $\dot{v} = \ddot{u}$:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ -pv + qu \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

or $\dot{U} = AU$, with $U = \begin{pmatrix} u \\ v \end{pmatrix}$.

$$\det(A - \lambda I) = -\lambda(-p-\lambda) + q = \lambda^2 + p\lambda + q$$

$$2\lambda = -p \pm \sqrt{p^2 - 4q}$$

Get 2 distinct roots unless $p^2 - 4q = 0$

In that case:
$$A = \begin{pmatrix} 0 & 1 \\ -p^2/4 & -p \end{pmatrix}$$

and $\lambda_1 = \lambda_2 = -p/2$.

Try to solve for eigenvectors: $(A - \lambda I)x = 0$

$$\begin{pmatrix} p/2 & 1 \\ -p^2/4 & -p/2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 0 \quad \Rightarrow \quad \begin{aligned} pu + 2v &= 0 \\ p(pu + 2v) &= 0 \end{aligned}$$

$v = -\frac{p}{2}u$ so $x = \begin{pmatrix} 1 \\ -p/2 \end{pmatrix}$ is an eigvec.

But there is no other eigenvector!

The matrix $A - \lambda I$ has a one-dim. nullspace.

Introduce then a <u>generalized eigenvector</u> y:

$$(A - \lambda I)y = x$$

$$\Rightarrow (A - \lambda I)^2 y = 0$$

Here take $y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Can then write $A = SJS^{-1}$

where $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ is a Jordan block

and $S = \begin{pmatrix} 1 & 1 \\ x & y \\ 1 & 1 \end{pmatrix}$.

The linear system $\dot{U} = AU$ then becomes:

$$\dot{U} = SJS^{-1}U$$

$$(S^{-1}U)^{\cdot} = J(S^{-1}U) \qquad \text{let } V = S^{-1}U$$

$$\dot{V} = JV \qquad V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\dot{v}_1 = \lambda v_1 + v_2$$

$$\dot{v}_2 = \lambda v_2 \Rightarrow v_2 \sim e^{\lambda t}$$

Then: $v_1 \sim t e^{\lambda t}$

Check: $\dot{v}_1 = e^{\lambda t} + \lambda t e^{\lambda t} = v_2 + \lambda v_1$.

Larger Jordan blocks give higher powers of t.

Extra: non-equal masses

$$M = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix}$$

$$M\ddot{x} + Kx = f$$

Could multiply by M^{-1}: $\ddot{x} + (M^{-1}K)x = M^{-1}f$

But then $M^{-1}K$ is not symmetric.

Instead, introduce generalized eigenvalue problem:

$$Ax = \lambda Bx \qquad A, B \text{ matrices.}$$

See eig(A,B) on Matlab

If A, B Hermitian, easy to show that $\lambda \in \mathbb{R}$, and $\bar{x}_1^T B x_2 = 0$ for $\lambda_1 \neq \lambda_2$.

If we solved for $(B^{-1}A)x = \lambda x$ we wouldn't know that $\lambda \in \mathbb{R}$.

In matrix form, $AS = BS\Lambda$.

With $A = K$, $B = M$, we have $A = MS\Lambda S^{-1}$.
So $M\ddot{x} + Kx = f$ becomes

$$M\ddot{x} + MS\Lambda S^{-1}x = f, \quad \text{let } y = S^{-1}x$$

Then $\ddot{y} + \Lambda y = S^{-1}M^{-1}f$.