Lecture 2: Minimum principles; Springs!

exercise: prove the following:

A if positive-definite iff all its principal minors are positive.

sub det along diagonal

Back to Strang:

$P(x) = \frac{1}{2}x^T A x - x^T b$ is minimized where $Ax = b$.

Min is $P(A^{-1}b) = -\frac{1}{2} b^T A^{-1} b$ (< 0! deep!)

proof: $\frac{1}{2}x^T A x - x^T b = \frac{1}{2}(x - A^{-1}b)^T A (x - A^{-1}b) - \frac{1}{2} b^T A^{-1} b$ $\blacksquare$

Sometimes $Ax = b$ doesn't have a solution, perhaps because A is not square.

Let's show:

(i) If A has linearly indep. columns, then $A^T A$ is symmetric +ve definite.

(ii) $A^T C A$ is also symm. +ve def. if C is.

proof: $x^T(A^TA)x = (Ax)^T(Ax)$

This is positive-def. if $=0$ only when $x=0$.

But this is exactly the statement of linear independence of columns: $x \neq 0 \Leftrightarrow Ax \neq 0$.

Same thing for $x^TA^TCAx = (Ax)^T C(Ax)$. ∎

Least-squares solution of $Ax=b$:

$Ax = b$ with linearly indep. columns.

If A is $m \times n$, this requires $m \geq n$.

Define error $e = b - Ax$.

$$\min_x \|e\|^2 = \min_x (b-Ax)^T(b-Ax)$$

$$= \min_x \left(x^TA^TAx - b^TAx - x^TA^Tb + b^Tb \right)$$

$$= 2 \min_x \underbrace{\left(\tfrac{1}{2}x^T(A^TA)x - x^TA^Tb \right)}_{P(x) \text{ with } A \to A^TA,\ b \to A^Tb} + b^Tb$$

Solution satisfies $A^TAx = A^Tb$

$$\min_x \|e\|^2 = -(A^Tb)^T(A^TA)^{-1}(A^Tb) + b^Tb$$

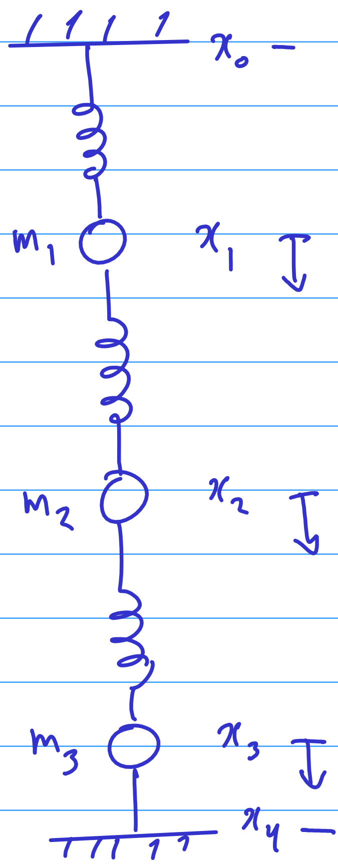
$$= b^T(I - A(A^TA)^{-1}A^T)b$$

Note that this is 0 if A^{-1} exists.

The solution to $A^TAx = A^Tb$ is the

least squares solution

Spring system: 4 springs in a line (equilibrium)



Vectors:

- x vector of nodal displacements
- e vector of elongation
- y vector of forces in the springs
- f vector of external forces on masses

They are related linearly:

$Ax = e, \quad Ce = y, \quad A^Ty = f$

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} = \begin{pmatrix} x_1 - 0 \\ x_2 - x_1 \\ x_3 - x_2 \\ 0 - x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad e = Ax$$

Now, force in a spring is proportional to elongation:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & c_3 & \\ & & & c_4 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} = Ce$$

spring constants
Hooke's law

Net force on a mass is zero at equilibrium:

$$f_i = y_i - y_{i+1}$$

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = A^T y$$

Why is it the transpose?

$$\underbrace{y^T e}_{\text{work done by stretching}} = y^T A x = \underbrace{f^T x}_{\text{external work done on nodes}}$$

Here $f_i = m_i g$ (gravitational force)

Putting it all together: $\underbrace{A^TCA}_{K\text{, stiffness matrix}} x = f$

symmetric +ve def.

This determines the equilibrium positions x.

Minimize

$$P(x) = \frac{1}{2} x^T A^T C A x - x^T f$$

the potential energy.

Here, $$A^TCA = \begin{pmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3+c_4 \end{pmatrix}$$

Note: +ve on diagonal, −ve off-diagonal

Can check that $(A^TCA)^{-1}$ has +ve entries.

This is important! $x = (A^TCA)^{-1} f$

Since $f_i > 0$, all masses must move down!

This is why $(A^TCA)^{-1}$ comes out with +ve entries.

Note: NOT the same as +ve definite!

Note: $(A^TCA)^{-1}$ is a full matrix (non-local solution)