Math 703 – Methods of Applied Math I

Lecture 1: Intro, Matrices & Pivoting

General info: www.math.wisc.edu/~jeanluc/703.html

Basic problem in numerical / applied math:

Solve $Ax = b$ Trivial?

Maybe not so trivial: computing A^{-1} is a bad idea!

Why? Speed, but more importantly storage

A sparse $\underset{\text{(usually)}}{\Longrightarrow}$ A^{-1} not sparse.

example:
(banded matrix)

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad A^{-1} = \text{full!}$$

Recall: Gaussian elimination: pivots

example: pivot
$$\begin{pmatrix} \boxed{2} & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = b$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & \boxed{3/2} & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & 0 \\ 0 & 0 & \boxed{4/3} & 0 \\ 0 & 0 & 1 & 5/4 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1 \\ 0 & 0 & 0 & \boxed{5/4} \end{pmatrix}$$

So here the pivots are $2, 3/2, 4/3, 5/4$

$$U = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1 \\ 0 & 0 & 0 & 5/4 \end{pmatrix}$$

Note that b was changed to c in this process.

$$Ax = b \Rightarrow Ux = c$$

What were the "steps" going from A to U?

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 2/3 & 1 & 0 \\ 0 & 0 & 3/4 & 1 \end{pmatrix} \begin{array}{l} \\ \leftarrow \frac{1}{2} \text{ row 1 from row 2} \\ \leftarrow \frac{2}{3} \text{ row 2 from row 3} \\ \leftarrow \frac{3}{4} \text{ row 3 from row 4} \end{array}$$

Hence, keeping track of those steps allows us to write:

$$\boxed{A = LU}$$ "LU decomposition"

To solve a linear system $Ax = b$

$\Rightarrow LUx = b$ Let $c = Ux$

Then: $Lc = b \Rightarrow$ solve for c, then get x

Solve two triangular systems: easy, by backsubstitution

Note: Gauss-Jordan involves turning A into I, thereby finding A^{-1}. Not usually needed.

When pivots $\neq 0$, can divide the rows of U by the pivots.

$$D = \begin{pmatrix} 2 & & & \\ & 3/2 & & \\ & & 4/3 & \\ & & & 5/4 \end{pmatrix}$$ matrix of pivots

$$A = LU = LD\underbrace{(D^{-1}U)}_{\text{upper-trig.}} = LD\tilde{U}$$

"LDU decomposition"

If A is also symmetric, $\tilde{U} = L^T$

$$A = LDL^T$$ "triple factorization"

Finally, if pivots > 0, then

$$A = (LD^{1/2})(LD^{1/2})^T = \bar{L}\bar{L}^T$$

"Cholesky factorization"

Let's do 2x2 matrices explicitly:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Take 2nd row $- \frac{c}{a}$ (first row):

$$L = \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{pmatrix}$$

Pivots are a, $d - \frac{bc}{a}$.

$$\tilde{U} = D^{-1} U = \begin{pmatrix} 1/a & 0 \\ 0 & \left(d - \frac{bc}{a}\right)^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix}$$

So LDU factorization is $LD\tilde{U}$.

If A is symmetric: $b = c$, so $\tilde{U} = L^T$.

When are pivots positive?

$$a > 0, \quad \frac{d - bc}{a} > 0 \quad \text{or} \quad \underbrace{ad - bc}_{\text{determinant!}} > 0$$

Let's explore the meaning of +ve pivots.

$\to$ symmetric $\to$ why?

Let $f = x^T A x$ be a quadratic form

When is $f > 0$ for all x with $|x| \neq 0$?

Simple example: $f = 2x_1^2 + 8x_1x_2 + 11x_2^2$

$$= (x_1 \ x_2)\begin{pmatrix} 2 & 4 \\ 4 & 11 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

complete squares $\longrightarrow$ $= 2(x_1 + 2x_2)^2 + 3x_2^2$

Hence, $f > 0$ whenever $|x| \neq 0$. $L = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

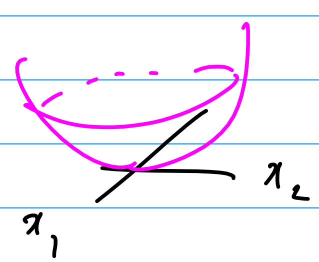
But now consider $A = LDL^T$.

$$x^T A x = x^T L D L^T x = (L^T x)^T D (L^T x)$$

$$= y^T D y = 2y_1^2 + 3y_2^2$$

So $x^T A x$ positive iff pivots > 0!

This is the same as saying $x^T A x$ has a minimum at $x = 0$



symmetric

Proposition: pivots are >0 iff $x^T A x > 0$ for $x \neq 0$.

proof: Assume pivots >0. Perform LU decomp of A.

First remove M_1 which contains multiples of the 1st row of A.

row 1: $\quad 1 \quad \times \quad [d \quad a_{12} \quad \cdots \quad a_{1n}]$

row 2: $\quad a_{21}/d \quad \times \quad$ "

$\vdots$

row n: $\quad a_{n1}/d \quad \times \quad$ "

$d \equiv a_{11}$ (pivot)

So $$M_1 = \underbrace{\begin{pmatrix} 1 \\ a_{21}/d \\ \vdots \\ a_{n1}/d \end{pmatrix}}_{l} \begin{pmatrix} d & a_{12} & \cdots & a_{1n} \end{pmatrix} = l\, d\, l^T$$

So $A - M_1$ now has the form $$\left(\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & & \\ 0 & & & \end{array}\right)$$

$n-1 \times n-1$ block

Now start with $A - M_1$ and repeat!

Find $M_2 = l_2 d_2 l_2^T$, etc.

Hence,

$$A = M_1 + \cdots + M_n = l_1 d_1 l_1^T + \cdots + l_n d_n l_n^T$$

So

$$x^T A x = x^T l_1 d_1 l_1^T x + \cdots + x^T l_n d_n l_n^T x$$
$$= (l_1^T x)^T d_1 (l_1^T x) + \cdots + (l_n^T x)^T d_n (l_n^T x)$$

sum of perfect squares! So > 0

Conversely, assume we've done the decomposition, with $x^T A x > 0$. Then $d_1 > 0$ by considering $x = (1\ 0\ 0 \cdots 0)$. But $B = A_{n-1,n-1}$ must also be +ve by considering $x = (0\ x_2 \cdots x_n)$, so $d_2 > 0$, etc.

$\blacksquare$