by thin filaments—the set M is connected. The branches follow a delicate combinatorial pattern. For values of c near the boundary of M, and especially down in "sea-horse valley" where a disk is attached, the Julia sets become wild and pleasurable. For c outside the Mandelbrot set, they break up into disconnected Cantor sets, or "Fatou dust".

We close with instructions for plotting a Julia set on a Macintosh (in 30 minutes?). Compute $x_1 = x_0^2 - y_0^2 - 1$ and $y_1 = 2x_0y_0$, then x_2 and y_2, stopping at x_{12} and y_{12}. Color the original x_0, y_0 white if $|x_{12}| > 2$ or $|y_{12}| > 2$. Do this for 129^2 points, with x_0 and y_0 equal to $-\frac{64}{32}, -\frac{63}{32}, \ldots, \frac{64}{32}$. For a new Julia set move c from -1. For the Mandelbrot set use a bigger machine.

EXERCISES

6.2.1 Solve the system $u_1' = -u_1, u_2' = -u_2$ and draw the paths of the point $u_1(t), u_2(t)$ starting from various initial values. This is the *stable star* produced by the matrix $A = -I$ with equal negative eigenvalues.

6.2.2 What types of critical points can $u' = Au$ have if
(1) A is symmetric positive definite
(2) A is symmetric negative definite
(3) A is skew-symmetric
(4) A is negative definite plus skew-symmetric (choose example).

6.2.3 Reduce $\theta'' + 2\theta' + \theta = 0$ to a system $u' = Au$ with $u_1 = \theta$ and $u_2 = \theta'$. A has equal negative eigenvalues but only one eigenvector, indicating a *stable improper node.* With $\theta = te^{-t}$ sketch the path of $(u_1, u_2) = (\theta, \theta')$ approaching the origin.

6.2.4 (a) Solve $u_1' = -u_2$, $u_2' = 4u_1$ starting from (1,0) to confirm that stability is neutral and the origin is a center.
(b) Find the orbit by eliminating time, leaving $du_2/du_1 = -4u_1/u_2$, and show that the circles in Fig. 6.9b become ellipses.

6.2.5 For the skew-symmetric "cross product equation"

$$u' = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = Au$$

(a) write out u_1', u_2', u_3' and confirm that $u_1'u_1 + u_2'u_2 + u_3'u_3 = 0$
(b) show that the energy $E = \frac{1}{2}(u_1^2 + u_2^2 + u_3^2)$ is constant
(c) find the eigenvalues of A.
Since $u' = u \times w$ the solution rotates around the fixed vector $w = (a,b,c)$.

6.2.6 If $a = 2u_1$, $b = 3u_2$, $c = 4u_3$ the equation above becomes nonlinear:

$$u_1' = u_2u_3, \quad u_2' = -2u_1u_3, \quad u_3' = u_1u_2.$$

(a) Show that u' is still perpendicular to u and $E = \text{constant}$

(b) Find the linearized matrix A at the stationary points $u^* = (1,0,0)$, $(0,1,0)$, and $(0,0,1)$ and decide if those points are stable, unstable, or neutral.

The equation describes the rotation of this book in the air. If you make it spin you will see which of its three axes is unstable; please catch the book.

6.2.7 Convert a 2 by 2 system $u' = Au$ into a single equation for u_1 by differentiating $u_1' = au_1 + bu_2$ and substituting $u_2' = cu_1 + du_2 = cu_1 + d(u_1' - au_1)/b$. How are the coefficients in the single equation connected to A?

6.2.8 Sketch the curves $E = \text{constant}$ in the phase planes for $\theta'' + \theta = 0$ and $u'' + u + u^3 = 0$, after multiplying by θ' and u' and integrating to find E.

6.2.9 If the linearized problem has a center then nonlinear stability is undecided, as in

$$\text{(a)}\quad \begin{aligned} u_1' &= u_2 + u_1(u_1^2 + u_2^2) \\ u_2' &= -u_1 + u_2(u_1^2 + u_2^2) \end{aligned} \qquad \text{(b)}\quad \begin{aligned} u_1' &= u_2 - u_1(u_1^2 + u_2^2) \\ u_2' &= -u_1 - u_2(u_1^2 + u_2^2) \end{aligned}$$

The right sides F are zero at $u^* = (0,0)$. Compute A at u^* and show it has a center. Then multiply the equations for u_1' by u_1 and the equations for u_2' by u_2, and add to find differential equations for $E = u_1^2 + u_2^2$. Show that (a) is unstable and (b) is stable; neither is neutrally stable.

6.2.10 If $c^2 > 4$ then Fig. 6.12 is wrong; the damped pendulum no longer spirals in to equilibrium. Identify the types of critical points and sketch the correct picture in the phase plane.

6.2.11 Solve the following equations and draw solution curves in the phase plane:

$$\text{(a)}\ \ \theta'' + \theta = 0 \qquad \text{(c)}\ \ \theta''/\theta' = -\theta'/\theta$$

$$\text{(b)}\ \ \theta'' - \theta' = 0 \qquad \text{(d)}\ \ \theta'' + (\theta')^2 = 0$$

6.2.12 With internal competition the predator-prey system might be

$$u_1' = u_1 - u_1^2 - bu_1u_2, \qquad u_2' = u_2 - u_2^2 + cu_1u_2$$

Find all equilibrium points and their stability (for $c < 1$ and $c > 1$). Which points make sense biologically?

6.2.13 According to Braun, reptiles, mammals, and plants on the island of Komodo have populations governed by

$$\begin{aligned} u_r' &= -au_r - bu_ru_m + cu_ru_p \\ u_m' &= -du_m + eu_ru_m \\ u_p' &= fu_p - gu_p^2 - hu_ru_p. \end{aligned}$$

Who is eating whom? Find all equilibrium solutions u^*.

6.2.14 If $u' = Au$ and A is skew-symmetric show that the energy $E = u^Tu$ is a constant. (The derivative of this inner product is $E' = u^Tu' + (u')^Tu$.)

6.2.15 If $u' = Au$ and $MA + A^T M$ is negative definite show that $E = u^T Mu$ is decreasing. This is the Liapounov approach—to find a symmetric positive definite M that yields decreasing energy and proves stability for nearby nonlinear equations.

6.2.16 (a) Find the type of critical point at $u^* = (0,0)$ for

$$u_1' = u_1 + u_2 - u_1(u_1^2 + u_2^2), \quad u_2' = -u_1 + u_2 - u_2(u_1^2 + u_2^2).$$

(b) Multiply the equations by u_1 and u_2 respectively and add to find an equation for $E = u_1^2 + u_2^2$.

(c) Show that $E' = 0$ for the special value $E = 1$, and sketch orbits spiralling out and in to this limit cycle $u_1^2 + u_2^2 = 1$.

6.2.17 From their trace and determinant, at what times t do the following matrices change type (bifurcation)?

$$A_1 = \begin{bmatrix} 1 & -1 \\ t & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 4-t \\ 1 & -2 \end{bmatrix} \quad A_3 = \begin{bmatrix} t & -1 \\ 1 & t \end{bmatrix}$$

6.2.18 Compute the solution to van der Pol's equation by finite differences and sketch the limit cycle for $c = \frac{1}{2}$.

6.2.19 (Epidemic theory). Suppose $u(t)$ people are healthy at time t and $v(t)$ are infected. If the latter become dead or otherwise immune at rate b and infection occurs at rate a, then $u' = -auv, \quad v' = auv - bv.$

(a) Show that $v' > 0$ if $u > b/a$, so the epidemic spreads.
(b) Show that $v' < 0$ if $u < b/a$, so the epidemic slows down. (It never starts if $u_0 < b/a$.)
(c) Show that $E = u + v - (b/a)\log u$ is constant during the epidemic.
(d) What is $v_{\max}$ (when $u = b/a$) in terms of u_0?

6.2.20 For freely falling bodies with $u = \frac{1}{2}gt^2 + u_0't + u_0$, sketch the curves $(u(t), u'(t))$ in the phase plane starting from three different initial values.

6.2.21 Invent a real function F such that $F(F(x)) = -x$.

6.2.22 Differentiate $G(u) = F(F(u))$ by the chain rule, and show that the slope G' has the same value $F'(U_1)F'(U_2)$ at both points $u = U_1$ and $u = U_2$ of a 2-cycle—for which $F(U_1) = U_2$ and $F(U_2) = U_1$.

6.2.23 On a computer with sound, assign different notes to subintervals of (0,1) so that you can hear the 2-cycles and 4-cycles of $u_{n+1} = au_n - au_n^2$.

6.2.24 Add periodic forcing to Duffing's equation, $u'' + u'/10 + u^3 = 12\cos t$, and display the solutions at many multiples of $t = 2\pi$ in the $u - u'$ phase plane.

6.2.25 Change the coefficient from 1.4 to 1.3 in Hénon's 2 by 2 system, and iterate 500 times. The limit is believed to have period 7.

6.2.26 The Cantor set is left when the middle thirds like $(\frac{1}{3}, \frac{2}{3})$, $(\frac{1}{9}, \frac{2}{9})$, $(\frac{7}{9}, \frac{8}{9})$, ... are removed from (0, 1). All numbers like 0.0200202 ... are still there—if they have no 1's when written in base 3. Where do the removed intervals $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$ have a 1?