displacement takes *less* force. Such a spring is unstable and stretches like a stick of gum that has reached the point of no return. The pendulum behaves like a soft spring, with $V'(u) = \sin u$ and with a period greater than $2\pi$. Other springs lead to a fascinating problem of *phase transition*, when $V'(u)$ increases again after decreasing. Then there are two stable phases surrounding the unstable one, and the choice becomes hard to predict.

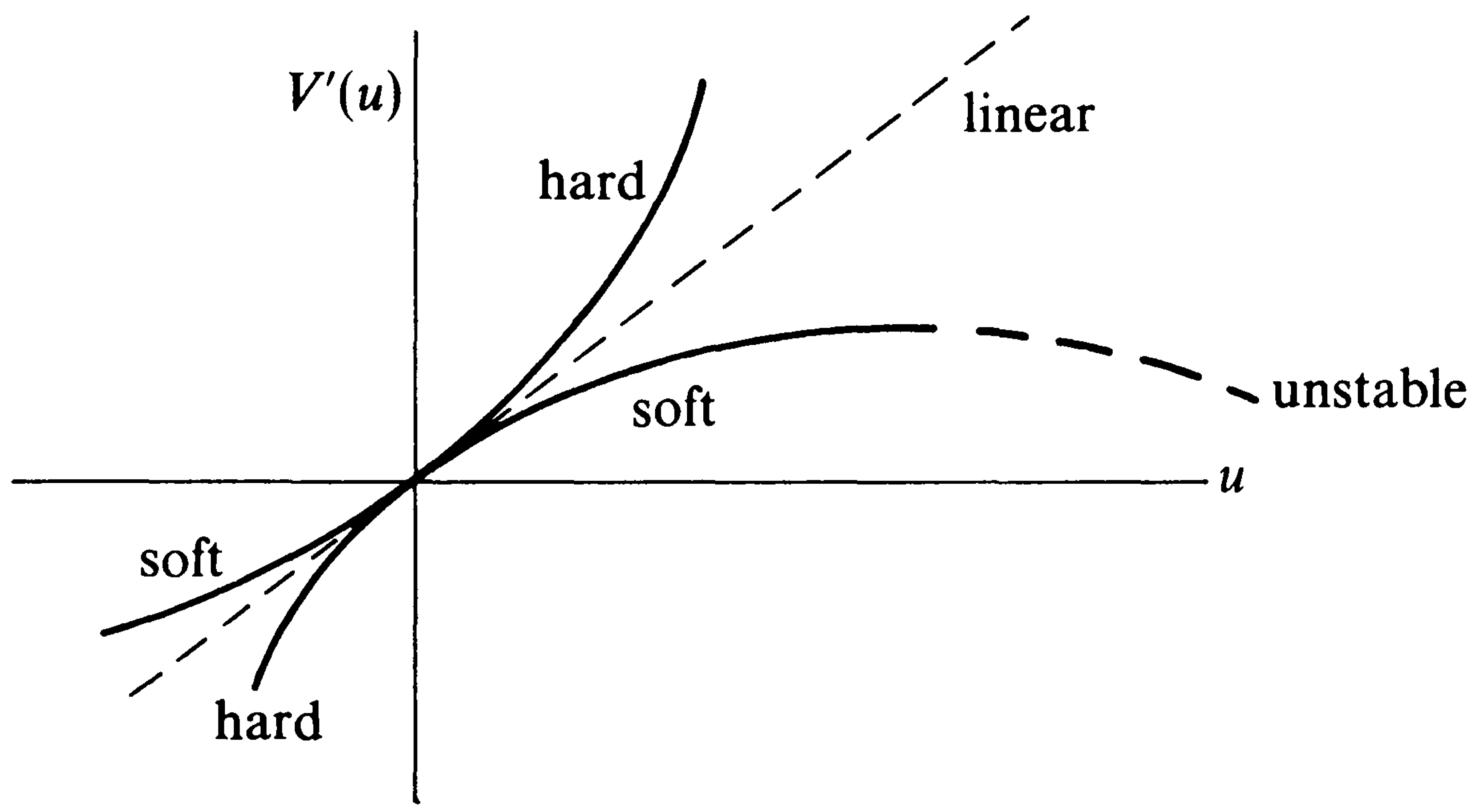


**Fig. 6.6.** Hard, soft, and linear springs.

The nonlinear picture is incomplete until damping is allowed. That makes an exact solution more difficult; we lose the integration that took us to $E$. The energy is not constant. But there is a beautiful method to study the oscillation, called the *phase plane*, which goes far beyond springs and electrical circuits. We will describe it after testing the equation for stability.

## EXERCISES

**6.1.1** Solve $u' + u = e^{2t}$, $u' + u = e^{i\omega t}$, $u' + u = e^{-t}$, and $u' + u = 1$ all with $u_0 = 5$. Which solutions go to a steady state $u_\infty$?

**6.1.2** If $u' + 2u =$ (delta function at $t = 1$) + $c$(delta function at $t = 4$), find the solution from equations (4–5). What choice of $c$ will turn the solution off, so that $u = 0$ for $t > 4$?

**6.1.3** Solve $du/dt = u^{1-k}$ with $u_0 = 1$, $k \neq 0$, by separating $u^{k-1}du$ from $dt$ and integrating. When does $u$ blow up if $k < 0$? Which of $u' = u^3$ and $u' = 1/u^3$ can be solved with $u_0 = 0$?

**6.1.4** Solve $u' - u \cos t = 1$ with $u_0 = 4$.

**6.1.5** Find the general solution to the separable equations

(a) $u' = -tu$ (b) $u' = -u/t$ (c) $uu' = \frac{1}{2}\cos t$

**6.1.6** The example $u' - u/t = 3t$ started from $u_0 = 0$ at $t = 1$. What is the integrating factor $e^{-h(t)}$? Multiply the equation by that factor, express the left side as an exact derivative, and integrate from $t = 1$ to find $u$.

**6.1.7** Change signs and solve $u' + u/t = 3t$ with $u(1) = 0$.

**6.1.8** Suppose a rumor starts with one person and spreads according to $u' = u(N - u)$. Find $u(t)$ for this logistic equation. At what time $T$ does the rumor reach half of the population ($u(T) = \frac{1}{2} N$)?

**6.1.9** Show by differentiating $v = u/(a - bu)$ that if $u' = au - bu^2$ then $v' = av$. The nonlinear logistic equation is linearized by a change of variable.

**6.1.10** Differentiate $u' = au - bu^2$ to show that $u'' = (au - bu^2)(a - 2bu)$. Where is the inflection point $u'' = 0$ in Fig. 6.1a, at which the curve changes from convex to concave?

**6.1.11** Find the solution with arbitrary constants $C$ and $D$ to

(a) $u'' - 9u = 0$ (b) $u'' - 5u' + 4u = 0$ (c) $u'' + 2u' + 5u = 0$

**6.1.12** Find an equation $u'' + pu' + qu = 0$ whose solutions are

(a) $e^t, e^{-t}$ (b) $\sin 2t, \cos 2t$ (c) $1, t$ (d) $e^{-t} \sin t, e^{-t} \cos t$

**6.1.13** (a) What damping constants $c$ in $\frac{1}{2} u'' + cu' + \frac{1}{2} u = 0$ produce overdamping, critical damping, underdamping, no damping, and negative damping?

(b) Find the exponents $\lambda_1, \lambda_2$ and solve with $u_0 = 2$ and $u_0' = -2c$. For which $c$ does $u(t) \to 0$?

**6.1.14** Find the undamped forced oscillation (21) for

(a) $u'' + u = \cos 2t$ (b) $u'' + 9u = \cos t$ (sketch $u$).

**6.1.15** Solve with $u_0 = 2$, $u_0' = 0$ and find the steady oscillation (23):

(a) $u'' + 2u' = \cos \omega t$ (b) $u'' + 2u' + 2u = \sin \omega t$

**6.1.16** What driving frequency $\omega$ will produce the largest amplitude $A$ in equation (24)? For small $R$ this is the "resonant frequency under damping."

**6.1.17** Show that (25) is the same as (24), with $\omega_0^2 = 1/LC$.

**6.1.18** (a) Solve $u'' + u' + u = t^2$ by assuming $u = A + Bt + Ct^2$.
(b) If the right side is $e^{-t}$, find $A$ in $u = Ae^{-t}$
(c) If the right side is $\cos t$, assume $u = A \cos t + B \sin t$.

*Note*: Exercise 18 uses the method of undetermined coefficients; 19 and 20 add a factor $t$ when $A$ or $Ae^{-t}$ or $A \cos t$ would fail.

**6.1.19** (a) Solve $u'' + u' = t^2$ with $u = At + Bt^2 + Ct^3$
(b) Solve $u'' + u' = e^{-t}$ with $u = Ate^{-t}$.

**6.1.20** For $u'' + u = \cos t$, show that $u = A \cos t + B \sin t$ fails to give a solution. This is resonance: solve with $u_0 = 0$ and $u_0' = 1$ and $u = C \cos t + D \sin t + At \cos t + Bt \sin t$.

**6.1.21** Find the energy $E(u)$ for the equations

(a) $u'' + \frac{1}{2}e^u - \frac{1}{2}e^{-u} = 0$ (b) $u'' + u - u^3 + u^5 = 0$.

If $u_0 = 0$ and $u_0' = 1$, what equation gives the amplitude $u_{max}$? Are these springs hard or soft?

**6.1.22** Suppose $a = 1$ but $b = -1$ in the logistic equation, giving cooperation instead of competition: $u' = u + u^2$. Solve for $u(t)$ if $u_0 = 1$. When does the population become infinite?