The factor $2\pi i$ that multiplies residues will cancel the factor $2\pi i$ in the inversion formula, and we recover the pulse $f = e^{-at}$.

EXAMPLE 2 $F(s) = \dfrac{1}{(a+s)^2}$ and $f(t) = \dfrac{1}{2\pi i}\displaystyle\int_{-i\infty}^{i\infty} \frac{e^{st}}{(a+s)^2}\, ds$

The path is closed by the same semicircle and the only novelty is the double pole. In principle the residue is the coefficient of $1/(a+s)$, which might seem to be zero for $e^{st}/(a+s)^2$. But remember that everything—including the exponential—must be expanded in powers of $a+s$:

$$\frac{e^{st}}{(a+s)^2} = \frac{e^{-at}e^{(a+s)t}}{(a+s)^2} = \frac{e^{-at}}{(a+s)^2}\left[1 + t(a+s) + \frac{t^2(a+s)^2}{2!} + \cdots\right].$$

The coefficient of $1/(a+s)$ is the residue te^{-at}, and this is $f(t)$. It also comes from the double pole formula (17), as the derivative of e^{zt} at $z = -a$.

The special case $a = 0$ is important. In the first example $F = 1/s$ corresponds to $f = 1$. In the second example $F = 1/s^2$ corresponds to $f = t$. In general $F = 1/s^{n+1}$ is the Laplace transform of $f = t^n/n!$ Those have poles at $s = 0$, on the path of integration. However a shift to the vertical line from $a - i\infty$ to $a + i\infty$, with the real number a chosen large enough, leaves all poles to the left. The semicircle will contain the poles, and the residues add to $f(t)$. That completes the inversion of F.

In Chapter 6 the Laplace transform solves initial-value problems in the same way that the Fourier transform solves boundary-value problems. All functions are decomposed into their frequency components; then the differential equation falls apart. You have to accept e^{-st} as if it were a harmonic (it is, but the frequency is imaginary) and superposition gives the answer. The final solution is a transient from Laplace plus a steady state from Fourier.

EXERCISES

4.5.1 If the line integral along C from P to Q depends only on P and Q, why does the case $Q = P$ (a closed loop) give

$$\int_C = \int_{-C} \quad \text{and} \quad \int_C = -\int_{-C} \quad \text{so} \quad \int_C = 0?$$

4.5.2 Compute the following integrals:

(a) $\int dz/z$ from 1 to i, the short way and long way on the circle $z = e^{i\theta}$

(b) $\int x\, dz$ around the unit circle, where $x = \cos\theta$ and $z = e^{i\theta}$—or alternatively where $x = \frac{1}{2}(z + z^{-1})$

(c) $\int dz/z$ around the circle $|z - 2i| = 1$

(d) $\int y^2\, dx + 2xy\, dy$ from $P = (0, 0)$ to $Q = (1, 1)$, noticing the exact differential (of what function U?)

(e) $\int dz/z$ for a path that winds three times around $z = 0$. (A simple closed curve winds only once.)

4.5.3 (a) Compute $\int dz/z^2$ around the circle $z = re^{i\theta}$, by substituting for z and dz and integrating directly.

(b) Despite the pole at $z = 0$ this integral is zero. What is the residue of $1/z^2$ at the pole?

(c) Why is $\int dz/z^2$ also zero around circles that are not centered at the origin?

4.5.4 Draw two circular disks in the complex plane, one not containing the origin and the other one centered at $z = 0$. In the first disk, mark the points where the absolute values $|z^2|$ and $|1/z^2|$ attain a maximum. In the second, mark the points where $|z^2|$ and the real part $x^2 - y^2$ and the imaginary part $2xy$ attain a maximum. Where does $|1/z^2|$ attain a maximum in the second disk?

4.5.5 If $f(z) = z^2$ on the circle $z = a + re^{i\theta}$ around the point a, substitute directly into Cauchy's integral formula (10) and show that it correctly gives $f(a) = a^2$.

4.5.6 Show that Cauchy's integral formula (10) for $f(a)$ reduces to the average value (11) at the center of a circle. What is the average value of e^z around the unit circle?

4.5.7 Find the location of the poles, and the residues, for

(a) $\dfrac{1}{z^2 - 4}$ (b) $\dfrac{z + 3}{z - 3}$ (c) $\dfrac{1}{(z^2 - 1)^2}$

(d) $\dfrac{e^z}{z^3}$ (e) $\dfrac{1}{1 - e^z}$ (f) $\dfrac{1}{\sin z}$

4.5.8 Evaluate the following integrals around the unit circle:

(a) $\displaystyle\int \frac{dz}{z^2 - 2z}$ (b) $\displaystyle\int \frac{e^z\, dz}{z^2}$ (c) $\displaystyle\int \frac{dz}{\sin z}$

4.5.9 By complex integration compute the real integrals

(a) $\displaystyle\int_0^{2\pi} \cos^4\theta\, d\theta$ (b) $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + \cos\theta},\quad a > 1$ (c) $\displaystyle\int_0^{2\pi} \cos^3\theta\, d\theta$

4.5.10 Find the poles above the real axis and evaluate

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^2}$ (b) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{4 + x^2}$ (c) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2 - 2x + 3}$

4.5.11 Find all the poles, branch points, and essential singularities of

(a) $\dfrac{1}{z^4 - 1}$ (b) $\dfrac{1}{\sin^2 z}$ (c) $\dfrac{1}{e^z - 1}$ (d) $\log(1 - z)$

(e) $\sqrt{4 - z^2}$ (f) $z \log z$ (g) $e^{2/z}$ (h) $\dfrac{e^z}{z^e}$

The point $z = \infty$ can be included by setting $w = 1/z$ and studying $w = 0$. Thus $z^3 = 1/w^3$ has a triple pole at $z=\infty$, $e^z=e^{1/w}$ has an essential singularity, and $\log z=-\log w$ has a branch point.

4.5.12 Find residues at $z=\pm i$ and use Fig. 4.22 to show that

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2}=\pi \quad\text{and}\quad \int_{-\infty}^{\infty}\frac{e^{ikx}dx}{1+x^2}=\pi e^{-|k|}.$$

4.5.13 Invert the following Laplace transforms to find $f(t)$ for $t\geq 0$:

(a) $F(s)=\dfrac{1}{s^2+1}$ (b) $F(s)=\dfrac{s}{s^2+1}$ (c) $F(s)=\dfrac{1}{(a+s)^3}$ (a triple pole)

CHAPTER 4 IN OUTLINE: ANALYTICAL METHODS

4.1 Fourier Series and Orthogonal Expansions—e^{ikx} as an eigenfunction of d/dx
The Fourier Coefficients—formulas for a_k, b_k, and c_k
Examples of Fourier Series—the coefficients for $f=\delta$ and $f=x$
Sine Series and Cosine Series—odd and even functions from 0 to π
Properties of Fourier Series—least squares approximations in Hilbert space
Solution of Laplace's Equation—$\sum a_k r^k \cos k\theta$ and Poisson's formula
Orthogonal Functions—the expansion $f=c_0T_0+c_1T_1+\cdots$
Bessel Functions—oscillations of a circular drum

4.2 Discrete Fourier Series and Convolution—the Fourier matrix F
The Discrete Transform for Arbitrary n—F^{-1} is $\bar{F}/n$
Discrete Convolutions—convolution rule and circulants $C=F\Lambda F^{-1}$
Signal Processing—convolution matrices $A_{ij}=a_{i-j}$

4.3 Fourier Integrals—the transform from f to $\hat{f}$ and its inverse
A List of Essential Transforms—delta functions, pulses, step functions
Energy and the Uncertainty Principle—energies $\int|f|^2=2\pi\int|\hat{f}|^2$
Derivatives, Integrals and Shifts—transforms of f', $\int f$, $f(x-d)$, $e^{ixd}f$
Convolution and Green's Functions—$\hat{u}=\hat{G}\hat{h}$ and fundamental solutions
Integral Equations—convolution kernels $K(x-y)$, solution by transform
The Sampling Theorem—band-limited f sampled at the Nyquist rate

4.4 Complex Variables and Conformal Mapping—the z-plane and w-plane
Analytic Functions and Laplace's Equation—Cauchy-Riemann equations
Conformal Mapping—boundary change preserving Laplace's equation
Important Conformal Mappings—e^z, $(az+b)/(cz+d)$, $\frac{1}{2}(z+z^{-1})$
Two-dimensional Fluid Flows—$f=u+is$: potential and stream function
Green's Functions and Electric Fields—superposition of point sources

4.5 Complex Integration—Cauchy's theorem $\int f(z)dz=0$
Cauchy's Integral Formula—maximum principle, $2\pi i f(a)=\int f(z)dz/(z-a)$
Singularities and Residues—real integrals by complex methods
Fourier Integrals and Laplace Integrals—transforms computed from poles