all columns. The response to a distributed source h is the sum or integral of responses to point sources δ.

If K is symmetric then so is its inverse G. The response at x_0, y_0 to an impulse at x, y is the same as the response at x, y to an impulse at x_0, y_0:

$$G(x, y; x_0, y_0) = G(x_0, y_0; x, y) \quad \text{corresponds to } G_{ij} = G_{ji}.$$

This is true of (20); exchanging z and z_0 has no effect on $\log|w|$. It will be true throughout our whole framework $K = A^TCA$. We emphasize that equations other than $u_{xx} + u_{yy} = h$ have Green's functions. G was found by conformal mapping in this special case; for $d^2u/dx^2 = h$ it is in the exercises, and in three dimensions it changes from $(\log r)/2\pi$ to $1/4\pi r$.

EXAMPLE The electric field intensity comes from the potential by

$$E = -\operatorname{grad} u = -\begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \\ \partial u/\partial z \end{bmatrix}.$$

The minus sign is normal in electrostatics but otherwise this agrees with $v = \operatorname{grad} u$ for fluids. In a region free of charge the divergence of E is zero and u satisfies Laplace's equation. If the charge density is ρ then $\operatorname{div} E = \rho$ (Maxwell's equation). A point charge corresponds to a delta function on the right side of the equation, and the potential is $u = 1/4\pi r$. A line of charges along the z-axis has potential $u = (\log r)/2\pi$.

The theory of electrostatics copies the theory of ideal fluids with one significant exception. On a solid surface in the fluid the boundary condition was $\partial u/\partial n = 0$; there is no flow into the obstacle. On a conducting surface in the field the boundary condition is $u = \text{constant}$; there is no potential difference (or charge would flow along the surface to remove it). It is the *tangential* component of E that vanishes, where before it was the *normal* component of v. The equipotentials in the fluid correspond to lines of force in the electric field. The stream function s corresponds to the negative of the electric potential. They are both constant along the boundary. Complex variables will hardly notice the difference, since this reversal of $u + is$ requires only multiplication by i.

EXERCISES

4.4.1 For the complex numbers $z = 1 + i$ and $w = 3 - 4i$,

(a) find their sum and product
(b) find their positions in the complex plane
(c) find the positions of their conjugates $\bar{z} = 1 - i$ and $\bar{w} = 3 + 4i$
(d) find their absolute values $|z|$ and $|w|$
(e) write z and $\bar{z}$ in polar form (z is $|z|e^{i\theta}$) by finding θ.

4.4.2 Find the real and imaginary parts of

(a) $z = e^{-2i\theta}$

(b) $z = \dfrac{1}{1+i}$ $\left(\text{multiply by } \dfrac{1-i}{1-i}\right)$

(c) $\log z = \log(re^{i\theta})$

(d) $i \log i \log \log i$

4.4.3 What can you say about

(a) the sum of a complex number z and its conjugate $\bar{z}$?

(b) the conjugate of a number $z = e^{i\theta}$ on the unit circle?

(c) the product of two numbers on the unit circle?

(d) the sum of two numbers on the unit circle?

(e) the suspicious formula $e^{2\pi i a} = (e^{2\pi i})^a = 1^a = 1$?

4.4.4 Find the absolute value (or modulus) $|z|$ if

(a) $z = e^i$

(b) $z = \dfrac{1}{3-4i}$

(c) $z = \dfrac{3+i}{3-i}$

(d) $z = (3+4i)^2$

(e) $z = e^{3+4i}$

4.4.5 Find the real and imaginary parts of the analytic functions

(a) $f = 1 + i(x+iy)$

(b) $f = e^{(x+iy)^2}$

(c) $\cos(x+iy) = \frac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)})$.

Verify that u and s satisfy the Cauchy-Riemann equations.

4.4.6 The derivative df/dz of an analytic function is also analytic; it still depends on the combination $z = x + iy$. Find df/dz if $f = 1 + z + z^2 + \cdots$ or $f = z^{1/2}$ (away from $z = 0$).

4.4.7 Are the following functions analytic?

(a) $f = |z|^2 = x^2 + y^2$

(b) $f = \operatorname{Re} z = x$

(c) $f = \sin z = \sin x \cosh y + i \cos x \sinh y$.

Can a function satisfy Laplace's equation without being analytic?

4.4.8 If $u(x, y) = x + 4y$, find its conjugate function $s(x, y)$ from the Cauchy-Riemann equations. If $s = (1 + x)y$, find u. If $u = x^2$, why does no s satisfy those equations?

4.4.9 Decompose $f = 1/z$ into $u + is$ by making the denominator real:

$$f = \frac{1}{x+iy} = \frac{1}{x+iy}\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2}.$$

Verify that u and s satisfy the Cauchy-Riemann equations. Are the curves $u = \text{constant}$ and $s = \text{constant}$ hyperbolas or ellipses (or neither)?

4.4.10 The Cauchy-Riemann equations in polar coordinates, where $z = re^{i\theta}$, must still come from the chain rule:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial r} = \frac{\partial f}{\partial z} e^{i\theta} \quad \text{and} \quad \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial \theta} = \frac{\partial f}{\partial z} ire^{i\theta}.$$

(a) Multiply the first by ir to find the relation between $\partial f/\partial r$ and $\partial f/\partial \theta$

(b) Substituting $f = u(r, \theta) + is(r, \theta)$ into that relation, find the Cauchy-Riemann equations connecting u and s

(c) Show that these equations are satisfied by the powers $f = z^n = r^n e^{in\theta}$, for which $u = r^n \cos n\theta$ and $s = r^n \sin n\theta$, and also by $u = \log r$ and $s = \theta$ (from $f = \log z$)

(d) Combine the Cauchy-Riemann equations in (b) into the polar coordinate form of Laplace's equation:

$$\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

4.4.11 The function $1/(1-z)$ has a singularity at $z = 1$, but around any other point a it admits the power series

$$\frac{1}{1-z} = \frac{1}{(1-a)-(z-a)} = \frac{1}{1-a}\left(1 + \frac{z-a}{1-a} + \left(\frac{z-a}{1-a}\right)^2 + \cdots\right).$$

This geometric series converges when the repeated factor $r = (z-a)/(1-a)$ has magnitude below 1. Sketch the regions in the complex plane given by $|r| < 1$ for the three cases $a = 0$, $a = 2$, $a = i$.

4.4.12 The following series are convergent for any $|z| < 1$:

$$-\log(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots \quad \text{and} \quad \frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + \cdots$$

Identify the term by term derivative of the first, and the term by term integral of the second. Where is the singularity that prevents convergence in a larger region like $|z| < 2$?

4.4.13 For the exponential mapping $w = e^z = e^{x+iy}$, show that

(1) each horizontal line $y = b$ is changed into a ray from the origin (at what angle with the horizontal?)

(2) each vertical line $x = c$ is changed into a circle around the origin (of what radius?)

4.4.14 If $z = 1 + iy$ show that $w = 1/z$ is on the circle $|w - \frac{1}{2}| = \frac{1}{2}$, in agreement with Fig. 4.14.

4.4.15 The equation of a circle $|z - z_0|^2 = R^2$ can be rewritten as

$$pz\bar{z} + qz + \overline{qz} + r = 0 \quad (p, r \text{ real}).$$

Substitute $w = 1/z$ and show that this gives a similar equation for a circle in the w-plane. Thus inversion maps circles to circles.

4.4.16 Show that the linear transformation $w = (az + b)/(cz + d)$ is the result of the three simple transformations

$$z \to z_1 = cz + d,\ z_1 \to z_2 = 1/z_1,\ z_2 \to w = \frac{a}{c} + \frac{bc - ad}{c} z_2.$$

Since all three take circles into circles (not necessarily centered at the origin) so does any linear transformation.

4.4.17 For the map $w = \frac{1}{2}(z + z^{-1})$ in Fig. 4.15, what happens to points $z = x > 1$ on the real axis? What happens to points $0 < x < 1$? What happens to the imaginary axis $z = iy$?

4.4.18 (a) For the same mapping let $z = re^{i\theta}$ and show that $w = X + iY$ has

$$X = \tfrac{1}{2}(r + r^{-1})\cos\theta \quad \text{and} \quad Y = \tfrac{1}{2}(r - r^{-1})\sin\theta.$$

(b) Using $\cos^2\theta + \sin^2\theta = 1$ find the equation of the ellipse in the $X - Y$ plane that comes from the circle $r = 2$ (and also from $r = \frac{1}{2}$) in the $x - y$ plane.

(c) Using $(r + r^{-1})^2 - (r - r^{-1})^2 = 4$ find the equation of the hyperbola in the $X - Y$ plane that comes from the ray $\theta = \pi/4$. Sketch it into Fig. 4.15; the hyperbola should be perpendicular to the ellipse since the ray was perpendicular to the circle in the $x - y$ plane.

4.4.19 Derive grad $u \cdot$ grad $s = 0$ from the Cauchy-Riemann equations. Since the equipotential curves are perpendicular to grad u and the streamlines are perpendicular to grad s, the equation grad $u \cdot$ grad $s = 0$ confirms that these curves are perpendicular.

4.4.20 Given $f = u + is$, suppose we take $s(x, y)$ as the potential instead of $u(x, y)$. For the flow with this potential, what is the stream function? What function $F(z)$ will produce this flow?

4.4.21 (a) With a delta function at $x = 0$ solve

$$\frac{d^2u}{dx^2} = \delta(x) \quad \text{for } -1 \le x \le 1, \text{ with } u(1) = u(-1) = 0.$$

Away from $x = 0$ the equation is $d^2u/dx^2 = 0$. Thus u is $ax + b$ on one side and $cx + d$ on the other; it is continuous at $x = 0$ but u' jumps by 1. Find a, b, c, d to obtain this one-dimensional Green's function (not $\log r$ as in 2D).

(b) With the delta function moved to $x = x_0$ solve the same problem. The solution is now the Green's function $G(x, x_0)$ for a source at $x = x_0$.

(c) Compute $u(x) = \int_{-1}^{1} G(x, x_0)dx_0$ and verify that it is the correct solution to $d^2u/dx^2 = 1$ with $u(1) = u(-1) = 0$.

(d) What would be the solution to $d^2u/dx^2 = h(x)$?

4.4.22 For the mapping $w = \sin z$ show how real points in the w-plane correspond to boundary points of a "blocked channel" above the interval from $z = -\pi/2$ to $z = \pi/2$. Sketch the streamlines in the z-plane that correspond to horizontal lines in the w-plane.

4.4.23 Solve Laplace's equation in the 45° wedge if the boundary condition is $u = 0$ on both sides $y = 0$ and $y = x$.

(a) Where does $F(z) = z^4$ map the wedge?

(b) Find a solution with zero boundary conditions other than $u \equiv 0$.