integrated, it gives the sampling theorem:

$$f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{n=-\infty}^{\infty} f(-n)\, e^{ink} \right) e^{ikx}\, dk$$

$$= \sum_{n=-\infty}^{\infty} f(-n) \left[\frac{e^{ik(x+n)}}{2\pi i(x+n)} \right]_{-\pi}^{\pi}$$

$$= \sum_{n=-\infty}^{\infty} f(-n) \frac{\sin \pi(x+n)}{\pi(x+n)} = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(x-n)}{\pi(x-n)}.$$

Reversing the sign of n at the last step has no effect on a sum from $-\infty$ to ∞. And if W is different from π, the same argument applies to a $2W$-periodic function—or we can rescale the x variable by π/W and the k variable by W/π, to complete the proof. Realistically we would sample 5 or 10 times in each period, and not just twice, to avoid being drowned by noise.

Band-limited functions are exactly what "band-pass filters" are designed to achieve. They multiply the transform $\hat{f}$ of the input signal by a function that is nearly $\hat{a} = 1$ for the frequencies to be kept and $\hat{a} = 0$ for the frequencies to be destroyed. Of course the filter does that by convolving the function. The convolution of f with $a = (\sin Wx)/\pi x$ multiplies $\hat{f}$ by $\hat{a}$ and leaves it limited to the band $-W < k < W$.

EXERCISES

4.3.1 Find the transform $\hat{g}$ of the one-sided ascending pulse

$$g(x) = e^{ax} \quad \text{for } x < 0, \quad g(x) = 0 \quad \text{for } x > 0.$$

4.3.2 Find the Fourier transforms (with $f = 0$ outside the ranges given) of

(a) $f(x) = 1$ for $0 < x < L$
(b) $f(x) = 1$ for $x < 0$
(c) $f(x) = \int_0^1 e^{ikx}\, dk$
(d) the finite wave train $f(x) = \sin x$ for $0 < x < 10\pi$

4.3.3 Find the inverse transforms of

(a) $\hat{f}(k) = \delta(k)$ (b) $\hat{f}(k) = e^{-|k|}$ (separate $k < 0$ from $k > 0$).

4.3.4 Apply Plancherel's formula $2\pi \int |f|^2 dx = \int |\hat{f}|^2 dk$ to

(1) the square pulse $f = 1$ for $-1 < x < 1$, to find $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2}\, dt$

(2) the even decaying pulse, to find $\displaystyle\int_{-\infty}^{\infty} \frac{dt}{(a^2 + t^2)^2}$.

Note The next three exercises involve $f=e^{-x^2/2}$ and its transform $\hat{f}=\sqrt{2\pi}\,e^{-k^2/2}$.

4.3.5 Verify Plancherel's energy equation for $f=\delta$ and $f=e^{-x^2/2}$. Infinite energy is allowed.

4.3.6 What are the half-widths W_x and W_k of the bell-shaped function $f=e^{-x^2/2}$ and its transform? Show that equality holds in the uncertainty principle.

4.3.7 What is the transform of $xe^{-x^2/2}$? What about $x^2e^{-x^2/2}$, using **4L**?

4.3.8 Show that the odd pulse (Example 5) is $-1/a$ times the derivative of the even pulse (Example 4). Therefore the transform of the odd pulse should be what factor times the transform of the even pulse?

4.3.9 The decaying pulse e^{-ax} has derivative $-ae^{-ax}$ (and 0 for $x<0$), so that differentiation seems to multiply its Fourier transform by $-a$ instead of ik. How can this be?

4.3.10 Solve the differential equation

$$\frac{du}{dx}+au=\delta(x)$$

by taking Fourier transforms to find $\hat{u}(k)$. What is the solution u (the Green's function for this equation)?

4.3.11 Take Fourier transforms of the unusual equation

$$(\text{integral of } u)-(\text{derivative of } u)=\delta$$

to find $\hat{u}$ (using **4L**). Do you recognize u?

4.3.12 The convolution $C=f*g$ of the decaying pulse and ascending pulse (Ex. 1) is

$$C(x)=\int_{-\infty}^{\infty} f(x-y)g(y)dy \text{ with transform } \hat{C}=\hat{f}\hat{g}=\frac{1}{a+ik}\frac{1}{a-ik}=\frac{1}{a^2+k^2}.$$

Find C by recognizing this transform and also by explicitly computing the integral.

4.3.13 The square pulse with $f=1$ for $-\frac{1}{2}<x<\frac{1}{2}$ has transform $\hat{f}=(2/k)\sin k/2$. Graph the "hat function" $h=f*f$ whose transform is $\hat{f}^2$. (The cubic B-spline is $h*h=f*f*f*f$ and its transform is $\hat{f}^4$.)

4.3.14 Show that the Fourier transform of gh is the convolution $\hat{g}*\hat{h}/2\pi$ by repeating the proof of the convolution rule—but with e^{+ikx} to produce the inverse transform.

4.3.15 The derivative of the delta function is the *doublet* δ'. It is a "distribution" concentrated at $x=0$ and from integration by parts it picks out not $f(0)$ but $-f'(0)$:

$$\int f(x)\,\delta'(x)dx=-\int f'(x)\,\delta(x)dx=-f'(0).$$

(a) Why should the Fourier transform of δ' be ik?
(b) What does the inverse formula (5) give for $\int ke^{ikx}dk$?
(c) Exchanging k and x, what is the Fourier transform of $f(x)=x$?

4.3.16 If $f(x)$ is an even function then the integrals for $x>0$ and $x<0$ combine into

$$\hat{f}(k)=\int_{-\infty}^{\infty} f(x)e^{-ikx}dx = 2\int_0^{\infty} f(x)\cos kx\, dx$$

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(k)e^{ikx}dk=\frac{1}{\pi}\int_0^{\infty}\hat{f}(k)\cos kx\, dx$$

Find $\hat{f}$ in this way for the even decaying pulse $e^{-a|x|}$. What are the corresponding formulas for sine transforms when f is odd?

4.3.17 If f is a line of delta functions explain why $\hat{f}$ is too:

$$\text{the transform of } f=\sum_{n=-\infty}^{\infty}\delta(x-2\pi n) \text{ is } \hat{f}=\sum_{n=-\infty}^{\infty}\delta(k-n).$$

The footnote after equation (13) may be useful.

4.3.18 (a) Why is $F(x)=\Sigma_{n=-\infty}^{\infty} f(x+2\pi n)$ a 2π-periodic function?

(b) Show that its Fourier coefficient $c_k=\frac{1}{2\pi}\int_{-\pi}^{\pi} Fe^{-ikx}dx$ equals $\hat{f}(k)/2\pi$.

(c) From $F(x)=\Sigma\, c_k e^{ikx}$ at $x=0$ find *Poisson's summation formula*:

$$\sum_{n=-\infty}^{\infty} f(2\pi n)=\sum_{k=-\infty}^{\infty}\hat{f}(k)/2\pi$$

4.3.19 If $u(x)=1$ then it is an eigenfunction for convolution: $k*1$ is a multiple of 1. Prove this directly and show that $k(0)$ is the multiple. The same argument for $u=e^{i\omega x}$ gave the eigenvalue $\hat{k}(\omega)$ in equation (20).

4.3.20 Another proof of positive definiteness when $\hat{k}(\omega)>0$ is to show that the quadratic form u^TKu is positive for every u. If K is a convolution then

$$u^TKu=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}k(x-y)u(y)u(x)dydx=\frac{1}{2\pi}\int_{\infty}^{\infty}\hat{k}(\omega)|\hat{u}(\omega)|^2d\omega>0.$$

Use the convolution rule on the y-integral and Plancherel's formula (9′) on the x-integral to establish this identity.

4.3.21 Apply Fourier transforms to $\int_{-\infty}^{\infty} e^{-|x-y|}u(y)dy-2u(x)=f(x)$ to show that the solution is $u=-\frac{1}{2}f+\frac{1}{2}g$, where g comes from integrating f twice. (Its transform is $\hat{g}=\hat{f}/(i\omega)^2$.) If $f=e^{-|x|}$ find u and verify that it solves the integral equation.

4.3.22 (a) If $f(x)=e^{i\omega x}$ confirm that the solution $u(x)$ given by (25) is $i\omega e^{i\omega x}/(1+i\omega)$ and that it solves the integral equation of Example 2.

(b) In the first integral in (25) identify the functions whose transforms are $1/(1+i\omega)$ and $i\omega\hat{f}(\omega)$. Then the second form of (25) comes from the convolution rule.

4.3.23 (a) Take Fourier transforms to find $\hat{u}(\omega)$ if

$$4\int_{y=-\infty}^{\infty} e^{-|x-y|}u(y)dy+u(x)=f(x),\ -\infty<x<\infty.$$

(b) Express $\left[\dfrac{8}{1+\omega^2}+1\right]^{-1}$ as $1-\dfrac{8}{\omega^2+9}$ and find its inverse transform g.

(c) Write u as a convolution $f*g$ by the convolution rule.

4.3.24 Add two more types of convolution to the table of eigenfunctions, frequencies, and eigenvalues:

(i) finite continuous: $\int_0^{2\pi} a(x-y)u(y)dy$ where a and u are 2π-periodic

(ii) one-sided discrete: $\sum_{j=-\infty}^{i} a_{i-j}u_j$.

4.3.25 Why does the sampling formula $\Sigma\, f(n) \sin \pi(x-n)/\pi(x-n)$ give the correct value $f(0)$ at $x=0$?

4.3.26 Suppose the Fourier transform of f is $\hat{f}(k)=1$ for $-\pi<k<\pi$, $\hat{f}(k)=0$ elsewhere. Check that the sampling theorem is correct.

4.3.27 Take Fourier transforms in the equation $d^4G/dx^4 - 2a^2 d^2G/dx^2 + a^4G = \delta$ to find the transform $\hat{G}$ of the fundamental solution. How would it be possible to find G?

4.3.28 What is $\delta * \delta$?

4.3.29 Suppose g is the mirror image of f, $g(x)=f(-x)$. Show from (4) that $\hat{g}(k)=\hat{f}(-k)$. If f is an even function (equal to its own mirror image, so that $f=g$) then so is $\hat{f}$.

4.3.30 Suppose g is a stretched version of f, $g(x)=f(ax)$. Show that $\hat{g}(k)=a^{-1}\hat{f}(k/a)$ and illustrate with the even pulse $f=e^{-|x|}$.

4.3.31 If $f=e^{-x^2/2}$ has transform $\hat{f}=\sqrt{2\pi}\; e^{-k^2/2}$, use the previous exercise to find the transform of $g=e^{-a^2x^2/2}$. Then show that $e^{-x^2/2} * e^{-x^2/2} = \sqrt{\pi}\; e^{-x^2/4}$, transforming the left side by the convolution rule (18) and the right side by the choice $a^2=\frac{1}{2}$.

Note on the transform $\hat{f}=\sqrt{2\pi}\; e^{-k^2/2}$: This is calculated in Exercise 6.4.4 and it comes also from the identity

$$\hat{f}(k)=\int_{-\infty}^{\infty} e^{-x^2/2}e^{-ikx}dx = e^{-k^2/2}\int_{-\infty}^{\infty} e^{-(x+ik)^2/2}\,dx.$$

The last integral is $\sqrt{2\pi}$ when $k=0$, and the change from $x+ik$ to x is justified by Cauchy's theorem in Section 4.5.

4.3.32 What is $\hat{f}$ if $f(x)=e^{5x}$ for $x\le 0$, $f(x)=e^{-3x}$ for $x\ge 0$?

4.3.33 Propose a definition of the two-dimensional Fourier transform. Given $f(x, y)$ what is $\hat{f}(k_1, k_2)$? Given $\hat{f}(k_1, k_2)$, what integral like (5) will invert the transform and recover $f(x, y)$?

4.3.34 Find the function $f(x)$ whose Fourier transform is $\hat{f}(k)=e^{-|k|}$.