integer n. Then the equation for B was (34):

$$rB'' + B' + \left(\lambda r - \frac{n^2}{r}\right) B = 0. \tag{40}$$

For $\lambda = 1$ this has a solution $J_n(r)$ which is finite at $r = 0$. That is the *Bessel function of order n*. (All other solutions blow up at $r = 0$; they involve Bessel functions of the second kind.) For every positive λ the solution is just rescaled to $J_n(\sqrt{\lambda}\, r)$. At $r = 1$ the boundary condition requires $J_n(\sqrt{\lambda}) = 0$; that picks out the eigenvalues. The products $A(\theta)B(r) = \cos n\theta\, J_n(\sqrt{\lambda_k}\, r)$ and $\sin n\theta\, J_n(\sqrt{\lambda_k}\, r)$ are the eigenfunctions. They give the shape of the drum in its pure oscillations, and Fig. 4.6 indicates roughly what they look like.

The simplest guide is the nodal lines along which the drum does not move. They are like the zeros of the sine function, where a violin string is still. For the drum we are in two dimensions and the eigenfunctions are $A(\theta)B(r)$. There is a nodal line from the center whenever $A = 0$ and a nodal circle whenever $B = 0$. For different values of n (the frequency in $\cos n\theta$) and k (the oscillation number in the r direction), the figure shows where the drumhead is motionless. The oscillations themselves are functions of time—they are solutions $A(\theta)B(r)e^{i\sqrt{\lambda}t}$ of the wave equation in a circle.

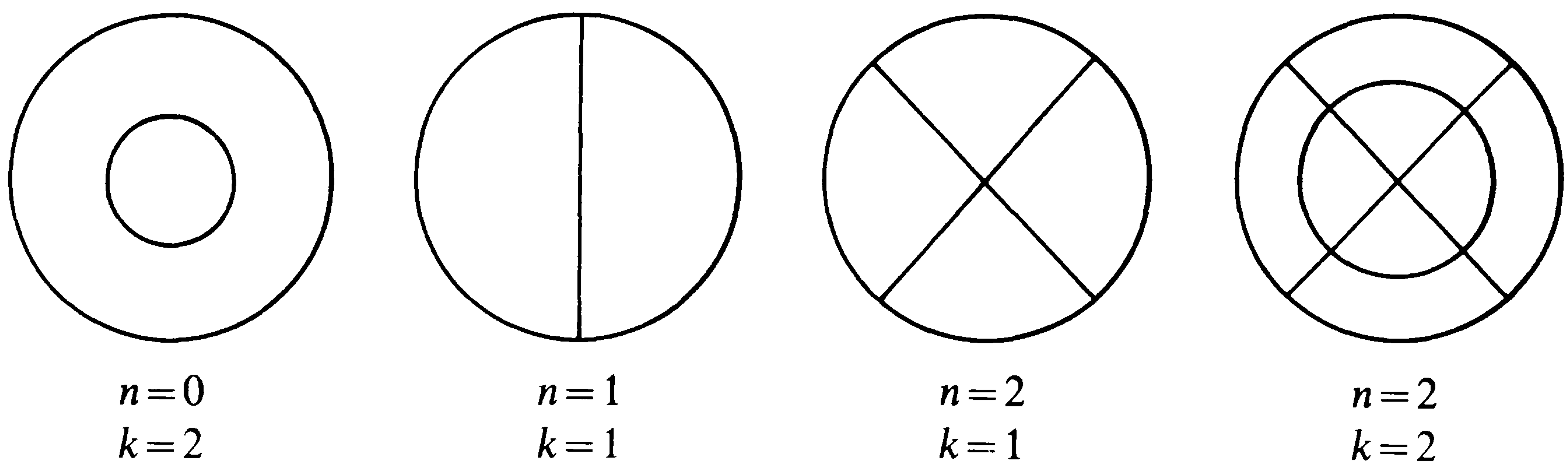

Fig. 4.6. Nodal lines of drum = zero lines of $A(\theta)B(r)$.

Finally we mention a problem that is unsolved as of Christmas 1984. ***Can you hear the shape of a drum***? If you know the eigenvalues λ, does that determine the boundary of the drumhead? I think the eigenvalues above, for a circle, do not occur for any other shape. But whether two different drums could sound the same, no one knows.

EXERCISES

4.1.1 Find the Fourier series on $-\pi < x < \pi$ for
(a) $f(x) = \sin^3 x$, an odd function
(b) $f(x) = |\sin x|$, an even function
(c) $f(x) = x^2$, integrating either $x^2 \cos kx$ or the sine series for $f = x$
(d) $f(x) = e^x$, using the complex form of the series.
What are the even and odd parts of $f(x) = e^x$ and $f(x) = e^{ix}$?

4.1.2 A square wave has $f(x) = -1$ on the left side $-\pi < x < 0$ and $f(x) = +1$ on the right side $0 < x < \pi$.

(1) Why are all the cosine coefficients $a_k = 0$?

(2) Find the sine series $\Sigma\, b_k \sin kx$ from equation (6).

4.1.3 Find this sine series for the square wave f in another way, by showing that

(a) $df/dx = 2\delta(x) - 2\delta(x+\pi)$ extended periodically

(b) $2\delta(x) - 2\delta(x+\pi) = \dfrac{4}{\pi}(\cos x + \cos 3x + \cdots)$ from (10)

Integrate each term to find the square wave f.

4.1.4 At $x = \pi/2$ the square wave equals 1. From the Fourier series at this point find the alternating sum that equals π:

$$\pi = 4(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} \cdots).$$

4.1.5 From Parseval's formula the square wave sine coefficients satisfy

$$\pi(b_1^2 + b_2^2 + \cdots) = \int_{-\pi}^{\pi} |f(x)|^2\, dx = \int_{-\pi}^{\pi} 1\, dx = 2\pi.$$

Derive another remarkable sum $\pi^2 = 8(1 + \frac{1}{9} + \frac{1}{25} + \cdots)$.

4.1.6 Around the unit circle suppose u is a square wave

$$u_0 = \begin{cases} +1 & \text{on the upper semicircle} \quad 0 < \theta < \pi \\ -1 & \text{on the lower semicircle} \quad -\pi < \theta < 0 \end{cases}$$

From the Fourier series for the square wave write down the Fourier series for u (the solution (21) to Laplace's equation). What is the value of u at the origin?

4.1.7 If a square pulse is centered at $x = 0$ to give

$$f(x) = 1 \quad \text{for} \quad |x| < \frac{\pi}{2}, \; f(x) = 0 \quad \text{for} \quad \frac{\pi}{2} < |x| < \pi,$$

draw its graph and find its Fourier coefficients a_k and b_k.

4.1.8 Suppose f has period T instead of 2π, so that $f(x) = f(x+T)$. Its graph from $-T/2$ to $T/2$ is repeated on each successive interval and its real and complex Fourier series are

$$f(x) = a_0 + a_1 \cos\frac{2\pi x}{T} + b_1 \sin\frac{2\pi x}{T} + \cdots = \sum_{-\infty}^{\infty} c_j e^{ij2\pi x/T}.$$

Multiplying by the right functions and integrating from $-T/2$ to $T/2$, find a_k, b_k, and c_k.

4.1.9 Establish by integration of x or otherwise the odd Fourier series

$$x(\pi - |x|) = \frac{8}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{27} + \frac{\sin 5x}{125} + \cdots\right), \; 0 < x < \pi.$$

4.1.10 What constant function is closest in the least square sense to $f=\cos^2 x$? What multiple of $\cos x$ is closest to $f=\cos^3 x$?

4.1.11 Sketch the graph and find the Fourier series of the even function $f=1-|x|/\pi$ (extended periodically) in either of two ways: integrate the square wave or compute (with $a_0=\frac{1}{2}$)

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx=\frac{2}{\pi}\int_0^{\pi}\left(1-\frac{x}{\pi}\right)\cos kx\,dx.$$

4.1.12 Sketch the 2π-periodic half wave with $f(x)=\sin x$ for $0<x<\pi$ and $f(x)=0$ for $-\pi<x<0$. Find its Fourier series.

4.1.13 Integrate the left side of (16) to find Bessel's inequality for the squares of the Fourier coefficients a_k and b_k.

4.1.14 (a) Find the lengths of the vectors $u=(1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\ldots)$ and $v=(1,\frac{1}{3},\frac{1}{9},\ldots)$ in Hilbert space and test the Schwarz inequality $|u^Tv|^2\le(u^Tu)(v^Tv)$.

(b) For the functions $f=1+\frac{1}{2}e^{ix}+\frac{1}{4}e^{2ix}+\cdots$ and $g=1+\frac{1}{3}e^{ix}+\frac{1}{9}e^{2ix}+\cdots$ use part (a) to find the numerical value of each term in

$$\left|\int_{-\pi}^{\pi}\bar{f}(x)g(x)dx\right|^2\le\int_{-\pi}^{\pi}|f(x)|^2\,dx\int_{-\pi}^{\pi}|g(x)|^2\,dx.$$

Substitute for f and g and use orthogonality (or Parseval).

4.1.15 In the solution to Laplace's equation with $u_0=\theta$ on the boundary, (26) is the imaginary part of $2(z-z^2/2+z^3/3\cdots)=2\log(1+z)$. Confirm that on the unit circle $z=e^{i\theta}$, the imaginary part of $2\log(1+z)$ agrees with θ.

4.1.16 If the boundary condition for Laplace's equation is $u_0=1$ for $0<\theta<\pi$ and $u_0=0$ for $-\pi<\theta<0$, find the Fourier series solution $u(r,\theta)$ inside the unit circle. What is u at the origin?

4.1.17 With boundary values $u_0(\theta)=1+\frac{1}{2}e^{i\theta}+\frac{1}{4}e^{2i\theta}+\cdots$, what is the Fourier series solution to Laplace's equation in the circle? Sum the series.

4.1.18 (a) Verify that the fraction in Poisson's formula satisfies Laplace's equation for each φ.

(b) What is the response $u(r,\theta)$ to an impulse at the point (0, 1) on the circle at the angle $\varphi=\pi/2$?

(c) If $u_0(\varphi)=1$ in the quarter-circle $0<\varphi<\pi/2$ and $u_0=0$ elsewhere, show that at points on the horizontal axis (and especially at the origin)

$$u(r,0)=\frac{1}{2}+\frac{1}{2\pi}\tan^{-1}\left(\frac{1-r^2}{-2r}\right)\quad\text{by using}$$

$$\int\frac{d\varphi}{b+c\cos\varphi}=\frac{1}{\sqrt{b^2-c^2}}\tan^{-1}\left(\frac{\sqrt{b^2-c^2}\sin\varphi}{c+b\cos\varphi}\right).$$

4.1.19 A plucked string goes linearly from $f(0)=0$ to $f(p)=1$ and back to $f(\pi)=0$. The linear part $f=x/p$ reaches to $x=p$, followed by $f=(\pi-x)/(\pi-p)$ to $x=\pi$. Sketch f as an odd function and find a plucking point p for which the second harmonic sin $2x$ will not be heard ($b_2 = 0$).

4.1.20 Show that $P_2 = x^2 - \frac{1}{3}$ is orthogonal to $P_0 = 1$ and $P_1 = x$ over the interval $-1 \le x \le 1$. Can you find the next Legendre polynomial by choosing c to make $x^3 - cx$ orthogonal to P_0, P_1, and P_2?

4.1.21 Using formula (30) with $f = |x|$, find the first 3 coefficients in the Legendre expansion $|x| = c_0P_0 + c_1P_1 + c_2P_2 + \cdots$. Sketch $|x|$ and $c_0P_0 + c_1P_1 + c_2P_2$ on the same graph for $-1 \le x \le 1$. To what functions is the difference of those graphs orthogonal?

4.1.22 If all orthogonal functions T_k are multiplied by 10 what happens to their coefficients c_k in (30)?

4.1.23 A function $f(x, y)$ of two variables in the square $-\pi < x, y < \pi$ can have the *double* Fourier series

$$f(x, y) = \sum\sum c_{jk} e^{ijx} e^{iky} \quad \text{(complex form with } -\infty < j, k < \infty)$$

$$f(x, y) = a_{00} + a_{10} \cos x + a_{01} \cos y + a_{11} \cos x \cos y + \dots \text{(even)}$$

$$f(x, y) = b_{11} \sin x \sin y + b_{21} \sin 2x \sin y + b_{12} \sin x \sin 2y + \dots \text{(odd)}$$

By multiplying by the right functions and integrating over the square, give formulas for c_{jk} and b_{jk}. (If f is neither even or odd its real series will also include all products cos kx sin lx and sin kx cos lx.)

4.1.24 Find the double Fourier coefficients c_{jk} if f in the square is

(a) a two-dimensional impulse $\delta(x, y)$: for any g, $\iint g\delta \, dxdy = g(0, 0)$
(b) a line of one-dimensional impulses $\delta(x)$: $\iint g\delta(x) \, dxdy = \int g(0, y) \, dy$
(c) $\cos^2 x \cos^2 y$

4.1.25 From the sine series for x in equation (12) and a similar series for y find the coefficient b_{kl} in the double sine series for $f = xy$.

4.1.26 If f has the double sine series $\Sigma\Sigma\, b_{kl} \sin kx \sin ly$, show that Poisson's equation $-u_{xx} - u_{yy} = f$ is solved by the double sine series $u = \Sigma\Sigma\, b_{kl} \sin kx \sin ly/(k^2 + l^2)$. This is the solution with $u = 0$ on the boundary of the square $-\pi < x, y < \pi$.

4.1.27 Find from $\partial E/\partial C_k = 0$ the coefficients C_k that minimize the error

$$E = \int [f - C_1T_1 - C_2T_2 \cdots]^2 \, w \, dx,$$

assuming that T_1, T_2, ... are orthogonal with weight w over the interval of integration. Compare with the coefficients c_k in equation (30).

4.1.28 Rodrigues' formula for the Legendre polynomials is $P_n = (2^n n!)^{-1} \, d^n(x^2 - 1)^n/dx^n$. Show that this gives $P_1 = x$ and $P_2 = (3x^2 - 1)/2$, and prove orthogonality by integrating $\int_{-1}^{1} P_2P_1 \, dx$ by parts. Why does the formula always produce a polynomial of degree n?

4.1.29 The polynomials $1, x, y, x^2 - y^2, 2xy, \dots$ solved Laplace's equation in two dimensions. Find five independent combinations of $x^2, y^2, z^2, xy, xz, yz$ that satisfy $u_{xx} + u_{yy} + u_{zz} = 0$. With spherical polynomials of all degrees we can match $u = u_0$ on the surface of a sphere.

4.1.30 Show that two eigenfunctions u_1 and u_2 of a Sturm-Liouville problem $(pu')' + qu + \lambda wu = 0$ are orthogonal with weight w. Multiply the equation for u_1 (with $\lambda = \lambda_1$) by u_2; multiply the equation for u_2 (with $\lambda_2 \neq \lambda_1$) by u_1; subtract and integrate over the interval. With zero boundary conditions integrate $u_2(pu_1')'$ and $u_1(pu_2')'$ by parts to show that $\int u_1 u_2 \, w dx = 0$.

4.1.31 Fit the Bessel equation (40) into the framework of a Sturm-Liouville equation $(pu')' + qu + \lambda wu = 0$. What are p, q, and w? What are they for the Legendre equation $(1 - x^2)P'' - 2xP' + \lambda P = 0$?

4.1.32 Show that the first Legendre polynomials $P_0 = 1$, $P_1 = \cos\varphi$, $P_2 = \cos^2\varphi - \frac{1}{3}$ are eigenfunctions of Laplace's equation $(wu_\varphi)_\varphi + w^{-1}u_{\theta\theta} = -\lambda wu$ with $w = \sin\varphi$ on the surface of a sphere. Find the eigenvalues λ of these *spherical harmonics*. The Legendre polynomials $P_n(\cos\varphi)$ are the eigenfunctions that are independent of the longitude θ.

4.1.33 Compare the $n!$ beneath r^n in the cosine series with $2^2 4^2 \cdots n^2$ in the Bessel series (36). Write the latter as $2^n[(n/2)!]^2$ and use Stirling's formula $n! \approx \sqrt{2\pi n}\, n^n e^{-n}$ to show that the ratio of these coefficients approaches $\sqrt{\pi n/2}$. They have the same alternating signs and the two series are very similar.

4.1.34 Substitute $B = \Sigma\, c_m r^m$ into Bessel's equation (40) and show from the analogue of (35) that λc_{m-2} must equal $(n^2 - m^2)c_m$. This recursion starts from $c_n = 1$ and successively finds $c_{n+2} = \lambda/(n^2 - (n-2)^2)$, $c_{n+4}, \ldots$ as the coefficients in a "Bessel function of order n:"

$$B_n(r) = r^n \left[1 + \frac{\lambda r^2}{n^2 - (n+2)^2} + \frac{\lambda^2 r^4}{(n^2 - (n+2)^2)(n^2 - (n+4)^2)} + \cdots \right]$$

$$= \frac{n!}{2^n} \sum_{k=0}^{\infty} \frac{(-1)^k (\sqrt{\lambda}/2)^{2k+n}}{k!(k+n)!}.$$

4.1.35 Where are the drum's nodal lines in Fig. 4.6 if $n = 1$, $k = 2$ or $n = 2$, $k = 3$?

4.1.36 Explain why the third Bessel eigenfunction $B = J_0(\sqrt{\lambda_3}\, r)$ is zero at $r = (\lambda_1/\lambda_3)^{1/2}$, $r = (\lambda_2/\lambda_3)^{1/2}$, and $r = 1$.