and light travels by the fastest route; action becomes time. If Planck's constant could go to zero, the deterministic principles of least action and least time would appear and the path would be not only probable but certain.

EXERCISES

3.6.1 What are the weak form and the strong form of the linear beam equation—the Euler equation for $P = \int [\frac{1}{2} c(u'')^2 - fu]dx$?

3.6.2 Minimizing $P = \int (u')^2 dx$ with $u(0) = a$ and $u(1) = b$ also leads to the straight line through these two points. Write down the weak form and the strong form.

3.6.3 Find the Euler equations (strong form) for

(a) $\int [(u')^2 + e^u]dx$ (b) $\int uu'\, dx$ (c) $\int x^2(u')^2 dx$

3.6.4 If $F(u,u')$ is independent of x, as in almost all our examples, show from the Euler equation and the chain rule that $H = u'\partial F/\partial u' - F$ is constant. This is dual to the fact that $\partial F/\partial u'$ is constant when F is independent of u.

3.6.5 If the speed is x the travel time is

$$T = \int_0^1 \frac{1}{x}\sqrt{1 + (u')^2}\, dx \quad \text{with} \quad u(0) = 0 \quad \text{and} \quad u(1) = 1.$$

(a) From the Euler equation what quantity is constant (Snell's law)?
(b) Can you integrate once more to find the optimal path $u(x)$?

3.6.6 With the constraints $u(0) = u(1) = 0$ and $\int u\, dx = A$, show that the minimum value of $P = \int (u')^2 dx$ is $12A^2$. Introduce a multiplier m, solve the Euler equation for u, and verify that $A = -m/24$. Then the derivative $dP/dA = 24A$ is $-m$ as the theory predicts.

3.6.7 For the shortest path constrained by $\int u\, dx = A$, what is unusual about the solution in Fig. 3.14 as A becomes large?

3.6.8 Suppose the constraint is $\int u\, dx \geq A$, with inequality allowed. Why does the solution remain a straight line as A becomes small? Where does the multiplier m remain?
Note: This is typical of inequality constraints: either the Euler equation is satisfied or the multiplier is zero.

3.6.9 Suppose the constrained problem is reversed, and we *maximize* the area $P = \int u\, dx$ subject to fixed length $l = \int \sqrt{1 + (u')^2}\, dx$, with $u(0) = a$ and $u(1) = b$.
(a) Form the Lagrangian and solve its Euler equation for u
(b) How is the multiplier M related to m in the text?
(c) When do the constraints eliminate all functions u?

3.6.10 Find by ordinary calculus the shortest broken-line path between (0,1) and (1,1) that goes first to the horizontal axis $y = 0$. Show that the best path treats this axis like a mirror: angle of incidence = angle of reflection.

3.6.11 The principle of maximum entropy selects the probability distribution that maximizes $H = -\int u \log u \, dx$. Introduce Lagrange multipliers for the constraints $\int u \, dx = 1$ and $\int xu \, dx = 1/a$, and find by differentiation an equation for u. On the interval $0 < x < \infty$ show that the most likely distribution is $u = ae^{-ax}$.

3.6.12 If the second moment $\int x^2 u \, dx$ is also known show that Gauss wins again: the maximizing u is the exponential of a quadratic. If only $\int u \, dx = 1$ is known, the most likely distribution is $u =$ constant. The *least* information comes when only one outcome is possible, say $u(6) = 1$, since $u \log u$ is then identically zero.

3.6.13 A path that climbs around a cylinder has $x = \cos\theta$, $y = \sin\theta$, $z = u(\theta)$:

$$\text{its length is } L = \int \sqrt{dx^2 + dy^2 + dz^2} = \int \sqrt{1 + (u')^2} \, d\theta.$$

Show that $u' =$ constant satisfies Euler's equation. What kind of path is $(x,y,z) = (\cos\theta, \sin\theta, c\theta)$?

3.6.14 Starting with the nonlinear equation $-u'' + \sin u = 0$, multiply by v and integrate the first term by parts to find the weak form. What integral P is minimized by u?

3.6.15 Find the Euler equations (strong form) for

(a) $$P(u) = \frac{1}{2} \iint \left[\left(\frac{\partial^2 u}{\partial x^2} \right)^2 + 2 \left(\frac{\partial^2 u}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 u}{\partial y^2} \right)^2 \right] dx \, dy$$

(b) $$P(u) = \frac{1}{2} \iint (y u_x^2 + u_y^2) \, dx \, dy$$ (c) $$E(u) = \int u \sqrt{1 + (u')^2} \, dx$$

(d) $$P(u) = \frac{1}{2} \iint (u_x^2 + u_y^2) \, dx \, dy \quad \text{with} \quad \iint u^2 dx \, dy = 1.$$

3.6.16 Show that the Euler equations for

$$\iint \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial y^2} \, dx \, dy \quad \text{and} \quad \iint \left(\frac{\partial^2 u}{\partial x \partial y} \right)^2 dx \, dy$$

are the same. (Presumably the two integrals are equal if the boundary conditions are zero.)

3.6.17 Sketch the graph of $p^2/2m + mgu =$ constant in the $u - p$ plane. It is an ellipse, parabola, or hyperbola? Mark the point where the ball reaches maximum height and begins to fall.

3.6.18 Draw a second spring and mass hanging from the first. If the masses are m_1, m_2 and the spring constants are c_1, c_2, the energy is

$$H = K + P = \frac{1}{2m_1} p_1^2 + \frac{1}{2m_2} p_2^2 + \frac{1}{2} c_1 u_1^2 + \frac{1}{2} c_2 (u_2 - u_1)^2.$$

Find the four Hamilton's equations $\partial H/\partial p_i = du_i/dt, \partial H/\partial u_i = -dp_i/dt$, and the matrix equation $Mu'' + Ku = 0$.

3.6.19 The Hamiltonian for a pendulum (with $u = \theta$) is $H = p^2/2m + mgl(1 - \cos u)$. Write out Hamilton's equations (21) and eliminate p to find the equation of a pendulum.

3.6.20 Verify that the energy $\frac{1}{2}e^TCe$ and the complementary energy $\frac{1}{2}w^TC^{-1}w$ are conjugate. As in equation (16), this means that $\frac{1}{2}w^TC^{-1}w = \max[e^Tw - \frac{1}{2}e^TCe]$.

CHAPTER 3 IN OUTLINE: EQUILIBRIUM IN THE CONTINUOUS CASE

3.1 One-dimensional Problems—analogies between discrete and continuous
A to A^T: Integration by Parts—the rule is $(Au)^Tw = u^T(A^Tw)$
Sturm-Liouville Problems—the solution to $-cu'' + qu = f$
Singular Perturbations—boundary layer as $c \to 0$

3.2 Differential Equations of Equilibrium—the Euler equation $\delta P/\delta u = 0$
Minimum Principles—essential and natural boundary conditions
Complementary Minimum Principle for w—minimize $Q = \int w^2/2c\, dx$
Fourth-order Equations—the beam equation $(cu'')'' = f$ has $A = (d/dx)^2$
Interpolation: Displacements and Slopes—four conditions on a cubic
Cubic Splines—continuous second derivatives at the nodes

3.3 Laplace's Equation and Potential Flow—$A^TAu = -\text{div grad}\, u = -u_{xx} - u_{yy}$
Boundary Conditions and Green's Formula—$(\text{grad})^T = -\text{div}$
Poisson's Equation—div (c grad u) $= f$
Minimum Principles—Laplace minimizes $P = \frac{1}{2}\int\int (u_x^2 + u_y^2)\, dx\, dy$

3.4 Vector Calculus in Three Dimensions—potentials and work $\int F \cdot dr$
Gradient, Divergence, and Curl—curl grad $u = 0$ and div curl $S = 0$
Electricity and Magnetism—Maxwell's equations, static and dynamic
Vector Calculus—interior integrals equal boundary integrals
Orthogonal Coordinate Systems—cylindrical and spherical scale factors

3.5 Equilibrium of Fluids and Solids—stress-strain law
Strain and Displacement—$e = \frac{1}{2}(J + J^T)$, examples of shear
Stress and Force—equilibrium div $\sigma + f = 0$
The Torsion of a Rod—warping functions and tensors
Fluid Mechanics—continuity equation and transport rule
Acceleration and Momentum Balance—$\rho Dv/Dt = \text{div}\, T$: perfect and viscous
Euler and Bernoulli Equations—$\frac{1}{2}v^2 + p/\rho = c$; vorticity and stream function
The Navier-Stokes Equations—similarity and the Reynolds number
The Stokes Equations—grad $p = \mu\nabla^2 v$ without acceleration

3.6 Calculus of Variations—the first variation $\delta P/\delta u = 0$: $\partial F/\partial u = (\partial F/\partial u')'$
Constrained Problems—Lagrange multiplier for $\int u\, dx = A$
Two-dimensional Problems—elliptic, parabolic, and hyperbolic
The Minimal Surface Problem—minimize $\int\int (1 + u_x^2 + u_y^2)^{1/2}\, dx\, dy$
Nonlinear Equations—variational form, weak form, strong form
The Energies F and F^*—Legendre transform yields complementary energy
Dynamics and Least Action—$Mu'' + Ku = 0$ and Hamilton's equations
Relativity and Quantum Mechanics—Einstein's energy $F = -mc(c^2 - v^2)^{1/2}$