That removes irrelevant edges. A large σ leaves only the big picture; a small σ allows a closer look. Pattern recognition is an ***inverse problem***—to recover the coloring book from the finished picture—like recovering the coefficients of a differential equation from its solutions.

EXERCISES

3.3.1 (a) Show that $u = x^3 - 3xy^2$ satisfies Laplace's equation.

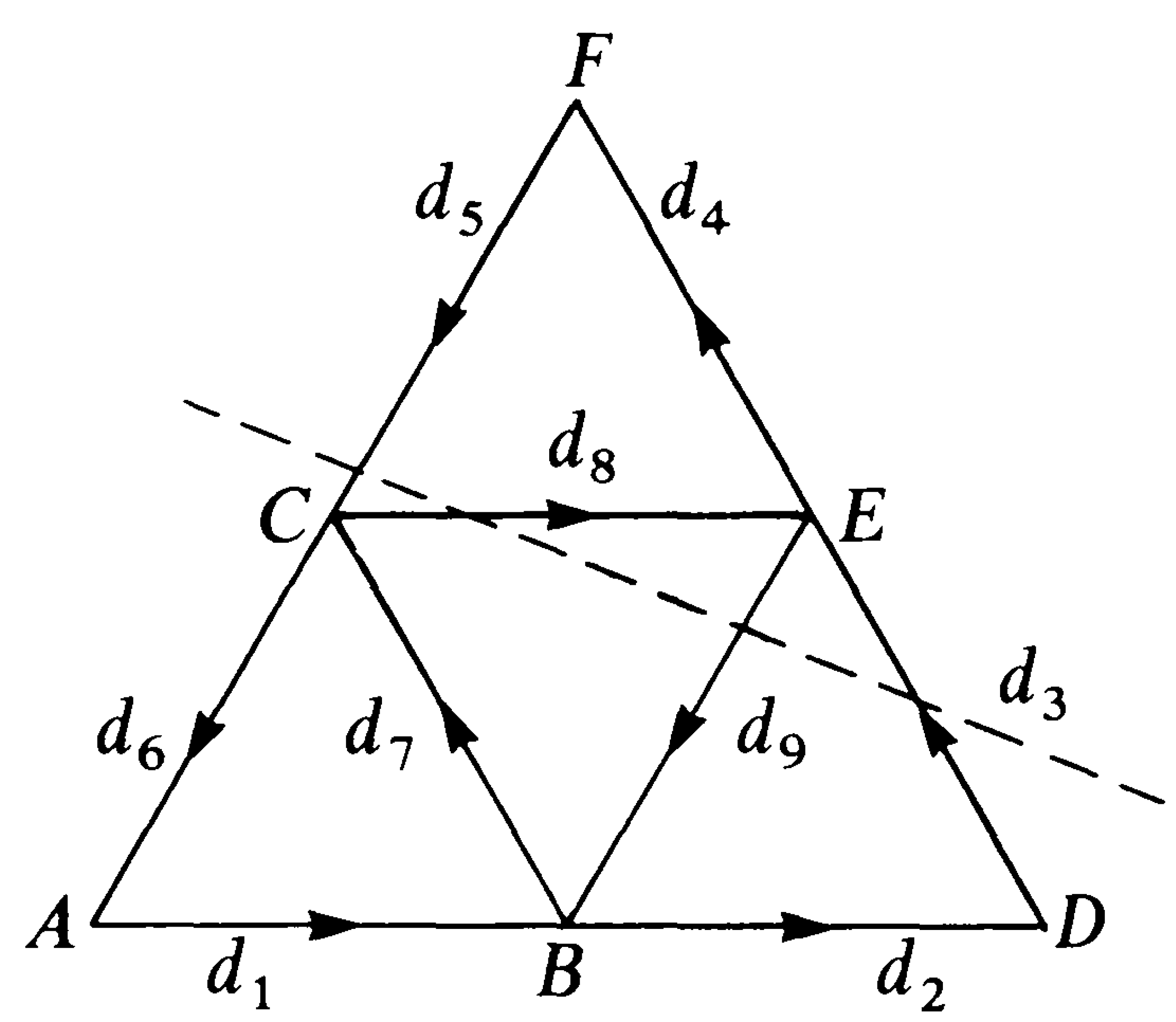
(b) Do the same for $s = 4x^3y - 4xy^3$, and explain where this comes in the list of polynomial solutions.

(c) Substitute $x = \cos\theta$ and $y = \sin\theta$ in s and simplify to an expression involving 4θ.

3.3.2 Verify that $u = e^x \cos y$ and $s = e^x \sin y$ both satisfy Laplace's equation, and sketch the equipotentials $u = $ constant and the streamlines $s = $ constant.

3.3.3 *Discrete divergence theorem*: Why is the flow across the "cut" in the figure equal to the sum of the flows from the individual nodes A,B,C,D? *Note*: This is true even if flows like $d_1 - d_6$ from nodes like A are nonzero. If the current law holds and each node has zero net flow, then the exercise says that the flow across every cut is zero.

3.3.4 *Discrete Stokes theorem*: Why is the voltage drop around the large triangle equal to the sum of the drops around the small triangles? *Note*: This is true even if voltage drops like $d_1 + d_7 + d_6$ around triangles like ABC are nonzero. If the voltage law holds and the drop around each small triangle is zero, then the exercise says that $d_1 + d_2 + d_3 + d_4 + d_5 + d_6 = 0$.



3.3.5 On a graph the analogue of the gradient is the edge-node incidence matrix A_0. The analogue of the curl is the loop-edge matrix R with a row for each independent loop and a column for each edge. Draw a graph with four nodes and six directed edges, write down A_0 and R, and confirm that $RA_0 = 0$ in analogy with curl grad $= 0$.

3.3.6 Why does the flow rate $w = (\partial s/\partial y,\ -\partial s/\partial x)$ satisfy $\operatorname{div} w = 0$ for any "stream function" $s(x,y)$?

3.3.7 If the density is $c = 1$ then

$$w = \begin{bmatrix} \frac{\partial s}{\partial y} \\ -\frac{\partial s}{\partial x} \end{bmatrix} \text{ is equal to } v = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix}.$$

Show from these Cauchy-Riemann equations $\partial u/\partial x = \partial s/\partial y$ and $\partial u/\partial y = -\partial s/\partial x$ that both u and s satisfy Laplace's equation.

3.3.8 The curves $u(x,y) = \text{constant}$ are orthogonal to the family $s(x,y) = \text{constant}$ if grad u is perpendicular to grad s. These gradient vectors are at right angles to the curves, which can be equipotentials and streamlines. Construct a suitable $s(x,y)$ from the geometry and verify $(\text{grad } u)^T (\text{grad } s) = \frac{\partial u}{\partial x}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial s}{\partial y} = 0$ if

(a) $u(x,y) = y$: equipotentials are parallel horizontal lines
(b) $u(x,y) = x - y$: equipotentials are parallel 45° lines
(c) $u(x,y) = \log(x^2 + y^2)^{1/2}$: equipotentials are concentric circles.

3.3.9 A differential equation like $dy/dx = f(x,y)$ gives a family of curves depending on the initial value $y(0)$, and $dy/dx = -1/f(x,y)$ gives the orthogonal curves. (The product of the slopes is -1, the usual condition for a right angle; the gradients are in the orthogonal directions $(1,f)$ and $(1,-1/f)$.) Solve $y' = -1/f$ for the second family if the first family is
(a) $y = e^x + \text{constant}$, from $dy/dx = e^x = f$
(b) $y = \frac{1}{2}x^2 + \text{constant}$, from $dy/dx = x = f$
(c) $xy = \text{constant}$, from $dy/dx = -y/x = f$.

3.3.10 In Stokes' law (8), let $v_1 = -y$ and $v_2 = 0$ to show that the area of S equals the line integral $-\int_C y\, dx$. Find the area of an ellipse ($x = a \cos t$, $y = b \sin t$, $x^2/a^2 + y^2/b^2 = 1$, $0 \le t \le 2\pi$).

3.3.11 By computing curl v, show that $v = (y^2,x^2)$ is not the gradient of any function u but that $v = (y^2,2xy)$ is such a gradient—and find u.

3.3.12 By computing div w, show that $w = (x^2, y^2)$ does not have the form $(\partial s/\partial y, -\partial s/\partial x)$ for any function s. Show that $w = (y^2, x^2)$ does have that form, and find the "stream function" s.

3.3.13 If $u = x^2$ in the square $S = \{-1 < x,y < 1\}$, verify the divergence theorem (11) when $w = \text{grad } u$:

$$\iint_S \text{div grad } u \, dx\, dy = \int_C n \cdot \text{grad } u \, ds.$$

If a different u satisfies Laplace's equation in S, what is the net flow through C?

3.3.14 What potential has the gradient $v = (u_x,u_y) = (2xy, x^2 - y^2)$? Sketch the equipotentials and streamlines for flow into a 30° wedge (Fig. 3.7 was 45°), and show that $v \cdot n = 0$ on the upper boundary $y = x/\sqrt{3}$. The streamlines have $s = xy^2 - \frac{1}{3}x^3 = \text{constant}$.

3.3.15 Solve Poisson's equation $u_{xx} + u_{yy} = 4$ by trial and error if $u = 0$ on the circle $x^2 + y^2 = 1$.

3.3.16 Find a quadratic solution to Laplace's equation if $u = 0$ on the axes $x = 0$ and $y = 0$ and $u = 3$ on the curve $xy = 1$.

3.3.17 Laplace's equation in polar coordinates is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

Show that $u = r \cos \theta + r^{-1} \cos \theta$ is a solution, and express it in terms of x and y. Find $v = (u_x, u_y)$ and verify that $v \cdot n = 0$ on the circle $x^2 + y^2 = 1$. This is the velocity of flow past a circle.

3.3.18 Show that $u = \log r$ satisfies Laplace's equation except at $r = 0$.

3.3.19 Suppose $\delta P/\delta u = \iint \left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} - fv \right] dx\, dy$. Use Green's formula, changing the u in (17) to v and changing w to grad u, to write

$$\frac{\delta P}{\delta u} = \iint_S v\,[?]\, dx\, dy + \int_C v\,[??]\, ds.$$

If this is zero for all v, find the differential equation and the natural boundary condition satisfied by u.