There is also a minimum principle. The spline has the smallest bending energy

$$P(u) = \int_0^n \left(\frac{d^2u}{dx^2}\right)^2 dx \tag{31}$$

among all functions with the correct heights $u_0, \ldots, u_n$ at the points $x = 0, \ldots, n$ (where the rings are). That gives the Euler equation $d^4u/dx^4 = 0$ away from the rings. In every interval u is a cubic, and the spline bends as little as possible. It is an excellent way to pass a curve through the prescribed points.

EXERCISES

3.2.1 For $P(u) = \frac{1}{2}\int_0^1 \left(\frac{d^2u}{dx^2}\right)^2 dx$, find the first variation $\delta P/\delta u$ from the linear term in $P(u + v)$.

3.2.2 What function $u(x)$ with $u(0) = 0$ and $u(1) = 0$ minimizes

$$P(u) = \int_0^1 \left[\frac{1}{2}\left(\frac{du}{dx}\right)^2 + x\,u(x)\right] dx?$$

3.2.3 What function $w(x)$ with $dw/dx = x$ (and unknown integration constant) minimizes

$$Q(w) = \int_0^1 \frac{w^2}{2}\, dx?$$

With no boundary condition on w this is dual to Ex. 3.2.2.

3.2.4 What functions u and w minimize P and Q with $dw/dx = x$ and $u(0) = w(1) = 0$? Verify the strong duality $-P = Q$.

3.2.5 With two conditions $w(0) = w(1) = 0$ show that no function satisfies $dw/dx = x$. With no conditions on u show that P has no minimum (except $-\infty$). This is the unstable case, unsupported and allowing rigid motions $u = \text{constant}$.

3.2.6 From the differential equation $-d/dx(c\,du/dx) = f$, derive the weak form (4) by multiplying by test functions v and integrating.

3.2.7 Show that $P(u) + Q(w) \geq 0$ for any admissible u and w:

$$\int_0^1 \left[\frac{c}{2}\left(\frac{du}{dx}\right)^2 - fu + \frac{1}{2c}w^2\right] dx \geq 0 \quad \text{when} \quad f = -\frac{dw}{dx} \quad \text{and} \quad u(0) = w(1) = 0.$$

3.2.8 Show that $P(u) + Q(w) = 0$ for the optimal u and w, which satisfy $w = cu'$.

3.2.9 If $u(0) = 0$ is changed to $u(0) = a$, this essential condition for $P(u)$ must again be a natural condition for $Q(w)$. That requires a change in Q:

$$Q(w) = \int_0^1 \frac{1}{2c} w^2 dx + aw(0) \quad \text{subject to} \quad -\frac{dw}{dx} = f \quad \text{and} \quad w(1) = 0.$$

Introduce the multiplier u for the constraint and show that the vanishing of the first variation $\delta L/\delta w$ does lead to $u(0) = a$. (The new term $aw(0)$ is the work done at the boundary; when w is perturbed to $w + V$, the difference $V(0)$ appears in $\delta L/\delta w$.)

3.2.10 If the ends of a beam are fixed (zero boundary conditions) and the force is $f = 1$ with $c = 1$, solve $d^4u/dx^4 = 1$ and then find M. Why does it have to be done in that order?

3.2.11 With simply supported ends, the boundary conditions make it possible to solve $M'' = f$ directly for M without going first to u. This is a statically determinate beam. Find M and u if $f = 1$ and $c = x$.

3.2.12 What is the shape of a uniform beam under zero force, $f = 0$ and $c = 1$, if $u(0) = u(1) = 0$ at the ends but $du/dx(0) = 1$ and $du/dx(1) = -1$? Sketch this shape.

3.2.13 The *step function* $s(x)$ is zero for $x \le 0$ and jumps to one for $x > 0$; its derivative is the delta function with $\delta(x) = 0$ for every $x \ne 0$ but $\int \delta(x)dx = 1$. To make this reasonable, consider the step s as a limit of functions s_n which have slope n between $x = 0$ and $x = 1/n$; elsewhere $s'_n = 0$. Compute $\int s'_n dx$ and $\int (s'_n)^2 dx$, and as $n \to \infty$ verify formally that $\int \delta = 1$ and $\int \delta^2 = \infty$.

3.2.14 What is the maximum height of the Hermite cubic $(x - 1)^2 x$ in Figure 3.5?

3.2.15 Prove that the coefficient matrix for spline interpolation in equation (29) is positive definite. One method of proof is to rewrite the quadratic

$$x^T A x = 2x_0^2 + 2x_0 x_1 + 4x_1^2 + 2x_1 x_2 + 4x_2^2 + \cdots$$

as a sum of many squares.

3.2.16 Which cubic spline has the values $u = 0,4,8,12$ at the nodes $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$? Show that the governing equation (29) is satisfied.

3.2.17 For the B-spline in Fig. 3.6, find the cubic in the second interval from its height and slope at $x = 1$ and $x = 2$.

3.2.18 (a) Find the cubic that replaces (24) if the right end of the interval is moved from $x = 1$ to $x = h$.

(b) Find the new form of (27) for this case, giving $d^2u/dx^2 = 0$ at $x = 0$.

Note: The new form of (28) at $x_1 = h$, if the second interval goes to $x_2 = h + H$, is a typical spline condition for unequally spaced points:

$$\frac{2s_1 + s_2}{H} + \frac{2s_1 + s_0}{h} = \frac{3}{H^2}(u_2 - u_1) + \frac{3}{h^2}(u_1 - u_0).$$

3.2.19 There are 16 coefficients in the four cubics between the points x_0, x_1, x_2, x_3, x_4. Describe in words the 16 conditions on a cubic spline; five of them are the prescribed heights u_0, u_1, u_2, u_3, u_4.