but you could never see the difference in Fig. 3.3. There is a ***boundary layer*** at each end in which all the action occurs. The layer reaches approximately to $x = 8\sqrt{c}$, which is enough for the special solution $u = 1 - e^{-x/\sqrt{c}}$ to climb from $u(0) = 0$ to $u = 1 - e^{-8}$. At that point it has virtually met the interior solution $U = 1$. Then a similar exponential at the other end connects $U = 1$ back to $u = 0$, in another boundary layer.

The perturbation is singular because the unperturbed solution $U = 1$ completely misses the boundary conditions. The leading term $-cu''$ is disappearing as c goes to zero, but it remains powerful inside the layer. Elsewhere the problem is calm.

(a) small q (b) small c

Fig. 3.3. A regular perturbation and a singular perturbation: $-cu'' + qu = 1$.

Note finally that a first derivative du/dx standing alone in (13) would have destroyed the whole framework. It corresponds to adding a skew-symmetric matrix to the existing A^TCA. Such a term does appear in fluid dynamics, and it illustrates the difference between diffusion and convection. Diffusion is symmetric and convection is not.

EXERCISES

3.1.1 For a bar with constant c but with decreasing $f = 1 - x$, find $w(x)$ and $u(x)$ as in equations (8–10).

3.1.2 For a hanging bar with constant f but weakening elasticity $c(x) = 1 - x$, find the displacement $u(x)$. The first step $w = (1 - x)f$ is the same as in (9), but there will be stretching even at $x = 1$ where there is no force. (The condition is $w = c\, du/dx = 0$ at the free end, and $c = 0$ allows $du/dx \neq 0$.)

3.1.3 Suppose a bar is free at both ends: $w(0) = w(1) = 0$. This allows rigid motion. Show that if $u(x)$ satisfies the differential equation and these boundary conditions, so does $u(x) + C$ for any constant C.

3.1.4 With the bar still free at both ends, what is the condition on the external force f in order that $-\dfrac{dw}{dx} = f(x)$, $w(0) = w(1) = 0$ has a solution? (Integrate both sides of the equation from 0 to 1.) This corresponds in the discrete case to solving $A_0^T y = f$; there is no solution for most f, because the left sides of the equations add to zero.

3.1.5 Find the displacement for an exponential force, $-u'' = e^x$ with $u(0) = u(1) = 0$.

Note that $A + Bx$ is the general solution to $-u'' = 0$; it can be added to any particular solution for the given f, and A and B can be adjusted to fit the boundary conditions.

3.1.6 Suppose the force f is constant but the elastic constant c jumps from $c = 1$ for $x \leq \frac{1}{2}$ to $c = 2$ for $x > \frac{1}{2}$. Solve $-dw/dx = f$ with $w(1) = 0$ as before, and then solve $c\, du/dx = w$ with $u(0) = 0$. Even if c jumps, the combination $w = c\, du/dx$ remains smooth.

3.1.7 Find the next term $W(x)$ in $u = \frac{1}{2}(x - x^2) + q(\frac{1}{12}x^3 - \frac{1}{24}x^4 - \frac{1}{24}x) + q^2 W + \cdots$. Choose W to match the q^2 terms in $-u'' + qu = 1$ and to satisfy $W(0) = W(1) = 0$.

3.1.8 For the negative value $q = -1$ show that $u = d_1 \cos x + d_2 \sin x - 1$ satisfies the differential equation $-u'' - u = 1$. The exponentials are e^{ix} and e^{-ix}, and they can be replaced by the sine and cosine.

3.1.9 If the condition at $x = 1$ were $u'(1) = 0$, why would no boundary layer be needed in Figure 3.3?

3.1.10 Verify that $u = d_1 e^{x/\sqrt{c}} + d_2 e^{-x/\sqrt{c}} + 1$ is an exact solution to $-cu'' + u = 1$. The condition $u = 0$ at $x = 0$ gives $d_1 + d_2 + 1 = 0$; find a similar equation from $u(1) = 0$ and solve for d_2. We expect $d_2 \approx -1$ to produce the boundary layer at $x = 0$.

3.1.11 What is the general solution to the constant-coefficient equation $-u'' + pu' = 0$? Try exponentials $u = e^{ax}$.

3.1.12 For $-u'' + pu' = 1$ with small p, find the regular perturbation pV by substituting $u = \frac{1}{2}(x - x^2) + pV$ and keeping the terms that are linear in p.

3.1.13 The solution to $-cu'' + u' = 1$ is $u = d_1 + d_2 e^{x/c} + x$. Find d_1 and d_2 if $u(0) = u(1) = 0$, and find their limits as $c \to 0$. The limit of u should satisfy $U' = 1$; which boundary condition does it keep and which end has a boundary layer?

3.1.14 Find the exponentials $u = e^{ax}$ that satisfy $-u'' + 5u' - 4u = 0$ and the combination that has $u(0) = 4$ and $u(1) = 4e$.

3.1.15 Solve the equation $-u'' = f$ with $u(0) = 0$ and $u'(1) = 0$ when f is a *delta function* at $x = \frac{1}{2}$. The impulse f is zero (and u is linear, $u = Ax + B$) except at $\frac{1}{2}$, where u' has a unit step down. The bar is stretched above $x = \frac{1}{2}$, then free.

3.1.16 Solve the same problem with $u(0) = u(1) = 0$, leading to the *Green's function* of page 351. The solution to $-u'' = \delta$ is again piecewise linear.

3.1.17 My class thinks that w in equation (9) should be $\int_0^x f\, dx + C$. But what constant of integration makes $w(1) = 0$?

Notes on the Dirac delta function (δ = unit impulse at $x = 0$)
Its integral from $-\infty$ to x is a *step function*: jump from 0 to 1 at $x = 0$
Second integral is a *ramp function* ($= x$ for $x > 0$; solution to $u'' = \delta$)
Third integral is a *quadratic spline* ($= \frac{1}{2}x^2$ for $x > 0$; jump in second derivative)
Fourth integral is a *cubic spline* ($= \frac{1}{6}x^3$ for $x > 0$; solution to $u'''' = \delta$, p. 177)
Its derivative δ' is a *doublet* (p. 327)
Delta function $\delta(x)\delta(y)$ in two dimensions: $\iint f(x, y)\, \delta(x)\delta(y)\, dx dy = f(0, 0)$
Defining property: $\int v(x)\delta dx = v(0)$ for every smooth function v