Summing over all bars, the complementary energy is the quadratic Q we met earlier:

$$Q(y) = \sum \frac{1}{2} \frac{y_i^2}{c_i} = \frac{1}{2} y^T C^{-1} y. \tag{12}$$

Therefore the energy principle discovered by Castigliano becomes exactly our theorem of duality. And the saddle point problem for $L = Q + x^T(A^T y - f)$ is known as the Hellinger–Reissner principle.

2H At equilibrium, the bar forces y minimize the complementary energy $Q(y)$ subject to $A^T y = f$. Furthermore the minimum values of P and Q satisfy $P_{\min} = -Q_{\min}$.

This is identical to the main result of Section 2.2 (with $b = 0$) after a sign change in x. There the quadratic was $P = \frac{1}{2} x^T A^T C A x + x^T f$; changing x to $-x$ reverses the linear term and produces the potential energy of a truss. The equations also reverse sign: mechanics has elongations $e = Ax$ instead of $e = -Ax$, $y = CAx$ instead of $y = -CAx$, and $f = A^T C A x$ instead of $f = -A^T C A x$. But the minimum of P is unchanged since $-x$ is as admissible as x.

The two principles are in perfect duality, but in practice one completely dominates the other. The ***displacement method*** (which minimizes P) is in constant use; the ***force method*** (which minimizes Q) is comparatively dormant. The reason can be found in the principles themselves. In the first, kinematic constraints like $x_j = 0$ are easy to impose. In the complementary principle we have to obey $A^T y = f$, and that is harder to do. It asks us to identify all the "redundancies," which are the solutions to $A^T y = 0$; they are the $m - n$ degrees of freedom in minimizing Q. For a small truss these self-stresses can be computed and added to a particular solution of $A^T y = f$. Codes for the nullspace are beginning to appear. But for a large truss or a discrete approximation to a continuous structure—as in the finite element method of Section 5.4, where thousands of unknowns are quite common—the displacement method seems to win.

EXERCISES

2.4.1 Write down m, N, r, and n for the three trusses in Fig. 2.10, and establish which is statically determinate, which is statically indeterminate, and which one has a mechanism. Describe the mechanism (the uncontrolled deformation).

2.4.2 With horizontal forces f_H^1 and f_H^2 pulling the upper nodes in Fig. 2.10a to the right, and vertical forces f_V^1 and f_V^2 pulling them up, write down the four equilibrium equations $A^T y = f$. Assuming the diagonal is at 30° and all $c_i = 1$, form the stiffness matrix $K = A^T C A = A^T A$.

2.4.3 With a single horizontal force f_H^1 applied to the upper left node in Fig. 2.10b, and the diagonal still at 30°, find the four equations $A^T y = f$. Since A is square solve directly for y. What reactive forces are supplied by the supports?

2.4.4 For the truss in Fig. 2.10c, write down the equations $A^T y = f$ in three unknowns y_1, y_2, y_3 to balance the four external forces $f_H^1, f_H^2, f_V^1, f_V^2$. Under what condition on these forces will the equations have a solution (allowing the truss to avoid collapse)?

2.4.5 For example 1 in the text, from Fig. 2.12a, equilibrium at the left support gives

$$-\frac{1}{\sqrt{2}} y_1 = f_H^1 \qquad \text{(horizontal reaction)}$$

$$-\frac{1}{\sqrt{2}} y_1 = f_V^1 \qquad \text{(vertical reaction)}$$

What are the corresponding equations at the second support? These four equations correspond to the four columns of A_0 eliminated by the fixed displacements $x_H^1 = x_V^1 = x_H^2 = x_V^2 = 0$ at the supports.

2.4.6 For example 3 (Fig. 2.12c) let the forces be $f_1 = f_2 = f_4 = f_6 = 0$, $f_3 = 1$, $f_5 = -1$. These satisfy the conditions for no rigid motion. Write down directly the solution to the 6 equations in the text for y_1, y_2, y_3.

2.4.7 In example 4 with a mechanism, what forces f_H and f_V at the lower node would make it possible to solve the three equations $A^T y = f$? F still acts horizontally at the roller.

2.4.8 With the bridge in Fig. 2.10a on top of the one in Fig. 2.10c (the supports remain only at the bottom) show that $m = n = 8$ but there is still a mechanism. What force would make this ladder collapse?

2.4.9 Sketch a six-sided truss with fixed supports at two opposite vertices. Will one diagonal crossbar between free nodes make it stable, or what is the mechanism? What are m and n? What if a second crossbar is added?

2.4.10 If we create a new node in Fig. 2.10a where the diagonals cross, is the resulting truss statically determinate or indeterminate?

2.4.11 In continuum mechanics, work is the product of stress and strain integrated over the structure: $W = \int \sigma\varepsilon \, dV$. If a bar has uniform stress $\sigma = y/A$ and uniform strain $\varepsilon = e/L$, show by integrating over the volume of the bar that $W = ye$. Then the sum over all bars is $W_{\text{total}} = y^T e$; show that this equals $f^T x$.

2.4.12 At the equilibrium $x = K^{-1} f$, show that the strain energy U (the quadratic term in P) equals $-P_{\min}$, and therefore $U = Q_{\min}$.

2.4.13 The "stiffness coefficients" k_{ij} in K give the forces f_i corresponding to a unit displacement $x_j = 1$, since $Kx = f$. What are the "flexibility coefficients" that give the displacements x_i caused by a unit force $f_j = 1$?

2.4.14 At equilibrium, where $x = K^{-1}f$, the terms in the potential energy $P(x)$ are $\frac{1}{2}x^TKx = \frac{1}{2}f^TK^{-1}f$ and $f^Tx = f^TK^{-1}f$. The internal strain energy and the external potential energy are not equal! Why not?

Note The point of virtual work is that, starting from x and making a small change v, the *changes* in internal and external terms are equal.

2.4.15 (a) Turn the square network of Exercise 2.3.14 into a truss. With the usual pin supports at the two nodes that were grounded, write down the 7 by 6 matrix in $e = Ax$.

(b) Which of the four types of truss is this?

(c) What is the rank of A and what are the solutions to $Ax = 0$?

(d) What are the solutions to $A^Ty = 0$?

2.4.16 Suppose a truss consists of *one bar* at an angle θ with the horizontal. Sketch forces f_1 and f_2 at the upper end, acting in the positive x and y directions, and corresponding forces f_3 and f_4 at the lower end. Write down the 1 by 4 matrix A_0, the 4 by 1 matrix A_0^T, and the 4 by 4 matrix $A_0^TCA_0$. For which forces can the equation $A_0^Ty = f$ be solved?

2.4.17 For *networks*, a typical row of $A_0^TCA_0$ (say row 1) is described on page 92: The diagonal entry is Σc_i, including all edges into node 1, and each $-c_i$ appears along the row. It is in column k if edge i connects nodes 1 and k. (A^TCA is the same with the grounded row and column removed.) The problem is to describe $A_0^TCA_0$ for *trusses*, and the idea is to put together the special $A_0^TCA_0$ found in the previous exercise (*a 4 by 4 matrix for each bar*).

(a) Suppose bar i goes at angle θ_i from node 1 to node k. By assembling the $A_0^TCA_0$ for each bar, show how the 2 by 2 upper left corner of $A_0^TCA_0$ contains

$$\begin{bmatrix} \Sigma\, c_i \cos^2\theta_i & \Sigma\, c_i \cos\theta_i \sin\theta_i \\ \Sigma\, c_i \cos\theta_i \sin\theta_i & \Sigma\, c_i \sin^2\theta_i \end{bmatrix}$$

(b) Where do those terms appear (with minus signs) in the first two rows? All rows of $A_0^TCA_0$ add to zero.

2.4.18 This is another approach to $A_0^TCA_0$ for trusses. The first column of A_0 contain $\cos\theta_i$ in row k, if bar i goes at angle θ_i from node 1 to node k. The second column contains $\sin\theta_i$. Multiply out $A_0^TCA_0$ to find its 2 by 2 upper left corner.

2.4.19 Sketch a square truss with horizontal forces f_1, f_3, f_5, f_7 and vertical forces f_2, f_4, f_6, f_8 at the nodes, numbered clockwise.

(a) Write down A_0 and $A_0^TCA_0$.

(b) There should be $8 - 4 = 4$ independent solutions of $A_0x = 0$. Describe or draw four movements x of the truss (rigid motion or mechanism?) that produce no stretching.

(c) From combinations of the 8 equations $A_0^Ty = f$, show that x^Tf must be zero for the four movements x of part (b). *For equilibrium, the force f must not activate the instabilities x.*