Its length squared is $6^2 + 3^2 = 45$. The other side of the rectangle is $y = (-1, 2)$, whose length squared is $(-1)^2 + 2^2 = 5$. Then $45 + 5$ agrees with $\|b\|^2 = 5^2 + 5^2 = 50$, confirming duality.

2D The column space S of a rectangular matrix A is orthogonal to the nullspace T of A^T. The projection of any vector b onto S is

$$Ax = A(A^TA)^{-1}A^Tb = Pb. \tag{17}$$

The projection of b onto T is $y = b - Ax$. The duality between them is equivalent to Pythagoras' law

$$\|Ax\|^2 + \|y\|^2 = \|b\|^2. \tag{18}$$

EXERCISES

2.2.1 Minimize $Q = \frac{1}{2}(y_1^2 + \frac{1}{3}y_2^2)$ subject to $y_1 + y_2 = 8$ in two ways:

(a) Solve $\partial L/\partial y = 0$, $\partial L/\partial x = 0$ for the Lagrangian $L = Q + x_1(y_1 + y_2 - 8)$.
(b) Solve the equilibrium equations (with $b = 0$) for x and y.

What is the optimal y, and what is the minimum of Q? What is the dual quadratic $-P(x)$, and where is it maximized?

2.2.2 Find the nearest point to the origin on the plane $y_1 + y_2 + \dots + y_m = 1$ by solving for y_1, substituting into $Q = \frac{1}{2}(y_1^2 + \dots + y_m^2)$, and minimizing with respect to the other y's. Then solve the same problem with Lagrange multipliers.

Note We could try to solve $A^Ty = f$ for the first n y's in terms of $y_{n+1}, \dots, y_m$. Then substitution in Q leaves only these $m - n$ unknowns. The method fails if the flows $y_1, \dots, y_n$ contain a loop—which is avoided in Section 2.3 but is sometimes difficult.

2.2.3 Find the minimum by Lagrange multipliers of

(a) $Q = \frac{1}{2}(y_1^2 + y_2^2)$ subject to $y_1 - y_2 = 1$

(b) $Q = \frac{1}{2}(y_1^2 + y_2^2 + y_3^2)$ subject to $y_1 - y_2 = 1$, $y_2 - y_3 = 2$ (use x_1 and x_2)

(c) $Q = y_1^2 + y_1y_2 + y_2^2 + y_2y_3 + y_3^2 - y_3$ subject to $y_1 + y_2 = 2$

(d) $Q = \frac{1}{2}(y_1^2 + 2y_1y_2) - y_2$ subject to $y_1 + y_2 = 0$ (watch for maximum).

Find the corresponding $P(x)$ in parts (a) and (b), and maximize $-P(x)$.

2.2.4 Find the rectangle with corners at points $(\pm y_1, \pm y_2)$ on the ellipse $y_1^2 + 4y_2^2 = 1$, such that the perimeter $4y_1 + 4y_2$ is as large as possible.

2.2.5 In three dimensions (and without any formulas) how far is the origin from the line $y_2 = 1$, $y_3 = 1$? What plane through this line is farthest from the origin? Where is the saddle point y—the nearest point on the line and on this farthest plane?

2.2.6 The minimum distance to the surface $A^T y = f$ equals the maximum distance to the hyperplanes which ____
Complete this statement of duality.

2.2.7 How far is it from the origin (0, 0, 0) to the plane $y_1 + 2y_2 + 2y_3 = 18$? Write this constraint as $A^T y = 18$, and solve for y in

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 18 \end{bmatrix}.$$

2.2.8 The previous question brings together several parts of mathematics if you answer it more than once:

(i) The vector y to the nearest point on the plane must be on the perpendicular ray. Therefore y must be a multiple of (1, 2, 2). What multiple lies on the plane $y_1 + 2y_2 + 2y_3 = 18$? What is the length of this y?

(ii) Since $A^T = [1\ 2\ 2]$ has length $(1 + 4 + 4)^{1/2} = 3$, the Schwarz inequality for inner products gives

$$A^T y \le \|A\|\ \|y\| \quad \text{or} \quad 18 \le 3\|y\|.$$

What is the minimum possible length $\|y\|$? Conclusion: The distance to the plane $A^T y = f$ is $|f|/\|A\|$.

2.2.9 In the first example of duality—"the minimum distance to points equals the maximum distance to planes"—how do you know immediately that maximum $\le$ minimum? In other words explain *weak duality*: The distance to any plane through the line is not greater than the distance to any point on the line.

2.2.10 If $b = (15, 10)$ in the geometry example of Fig. 2.4, what are the optimal Ax and y and what are the lengths in $\|Ax\|^2 + \|y\|^2 = \|b\|^2$?

2.2.11 Find the projection matrix P onto the 45° line $x_1 = x_2$, which is the column space of $A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Verify that $P^2 = P = P^T$. What is the projection of the point $b = (0, 1)$ onto this line? What are the eigenvalues of P?

2.2.12 If Ax is on S and y is on T but $z = y + Ax - b$ is not zero, show that Ax and y miss duality by $z^T z$ exactly as in (4). In other words, if $A^T y = 0$ verify the "quadrilateral law"

$$\|b - Ax\|^2 + \|b - y\|^2 = \|b\|^2 + \|z\|^2.$$

2.2.13 Suppose $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$.

Compute $x = (A^T A)^{-1} A^T b$, $P = A(A^T A)^{-1} A^T$, and $Ax = Pb$. What is the dimension of T (the subspace $A^T y = 0$) and how far away is b?

2.2.14 For any projection matrix P, with $P = P^T = P^2$, verify that

$$\|Pb\|^2 + \|(I - P)b\|^2 = \|b\|^2.$$

2.2.15 Suppose S is an n-dimensional subspace of R^m, and P is the matrix that projects onto S. From the geometry rather than the formula $P = A(A^TA)^{-1}A^T$, find the vectors whose direction is the same after projection:

(a) Why is every vector in S (and every vector orthogonal to S) an eigenvector of P?
(b) What are the corresponding eigenvalues?
(c) What is the column space of P (the set of all possible Pb)?
(d) What is the rank of P (the dimension of the column space)?
(e) What is the determinant of P?

2.2.16. In m dimensions, how far is it from the origin to the hyperplane $x_1 + x_2 + \cdots + x_m = 1$? Which point on the plane is nearest to the origin?