ground, which puts it ahead of the off-diagonal sum. For Fig. 2.1, with six edges and three ungrounded nodes,

$$A^TCA = \begin{bmatrix} c_1 + c_2 + c_5 & -c_1 & -c_2 \\ -c_1 & c_1 + c_3 + c_4 & -c_3 \\ -c_2 & -c_3 & c_2 + c_3 + c_6 \end{bmatrix}. \tag{7}$$

The matrices A^TA and A^TCA are symmetric positive definite "M-matrices," mentioned in an earlier footnote (Section 1.4). Apart from zeros, they are negative off the diagonal and ***all entries in their inverses are positive.***

4. Every area of applied mathematics has its own interpretation of A, C, b, and f. Some problems also have their own choice of sign; each section will translate between the specific application and the common framework. In certain cases (for example structures) the sign of x is reversed, and so is the sign of A:

$$\begin{bmatrix} C^{-1} & -A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix} = \begin{bmatrix} b \\ f \end{bmatrix}. \tag{8}$$

It is now $+A^TCA$ which appears halfway through elimination, and therefore all the pivots will be positive. But the matrix in (8) is no longer symmetric.

5. Strictly speaking Fig. 2.1 represents a ***directed graph.*** It is a ***network*** when a number c_i is assigned to every edge.

6. It is remarkable how well the framework extends to problems that are *continuous* rather than discrete. Instead of a finite number of edges, the flow may fill a region in the plane. The unknowns change from vectors to functions, and the potential differences change to derivatives; they still produce the flow. It is governed by differential equations instead of matrix equations. Nevertheless the pattern in Fig. 2.2 still leads to A^TCA, and describes the equilibrium of the system—this is the theme of Chapter 3.

EXERCISES

2.1.1 For a graph with edges around a square and across one diagonal ($N = 4$ and $m = 5$), number the nodes and edges and write down the incidence matrix A_0. What is $A_0^TA_0$?

The next three exercises refer to the network with four nodes and six edges at the beginning of the section. The edge constants are $c_1, \ldots, c_6$.

2.1.2 (a) Compute the 4 by 4 matrices $A_0^TA_0$ and $A_0^TCA_0$ for the network in Fig. 2.1. Notice that like the original A_0, its columns add up to the zero column.

(b) Verify that removing the last row and column of $A_0^T C A_0$ leaves $A^T C A$ in equation (7). What is $A^T A$?

(c) Show that this $A^T A$ is positive definite by applying one of the tests in Chapter 1 (for example, compute the determinants or the pivots).

2.1.3 For the triangular network in Fig. 2.1, let $f_1 = f_2 = f_3 = 1$ and $f_4 = -3$. With $C = I$ and $b = 0$, solve the equilibrium equation $-A^T C A X = f$. (Note that f_4 and x_4 do not enter, because $x_4 = 0$ and the last column of A_0 was removed.) Solve also for y, and describe the flows through the network.

2.1.4 Suppose there are "batteries" $b_4 = b_5 = b_6 = 1$ on the inner edges of the network in Fig. 2.1. With $f = b_1 = b_2 = b_3 = 0$ and $C = I$, write down the 9 by 9 equilibrium system (4). Show that *there is no flow*: the solution has $y = 0$ (but not $x = 0$).

2.1.5 For a network with only $m = 3$ edges and $N = 3$ nodes, at the vertices of a triangle with arrows clockwise, write down A_0 and A. With $f = 0$, $b_1 = b_2 = b_3 = 1$, and $C = I$, find x and y.

2.1.6 Suppose a network has N nodes and every pair is connected by an edge. Find m, the number of edges.

2.1.7 Imagine an R by R network in the plane, with nodes at the $N = R^2$ points with integer coordinates between 1 and R.

(a) If horizontal and vertical edges make it into a network of unit squares find the number of edges. Show that the approximate ratio of m to N is 2 to 1.

(b) If the network also includes the diagonals of slope $+1$ in each square, adding $(R-1)^2$ new edges, show that the approximate ratio of m to N for this triangular mesh is 3 to 1.

2.1.8 Show that the particular matrix

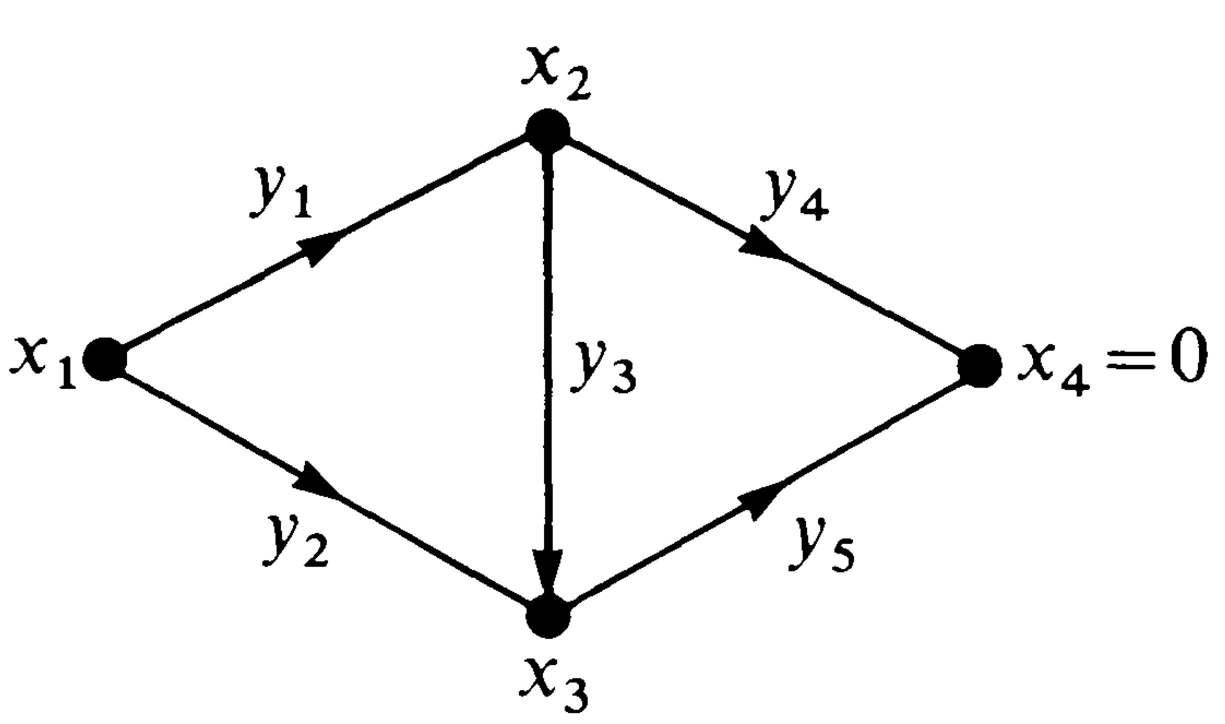
$$M = \begin{bmatrix} C^{-1} & A \\ A^T & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

is neither positive definite nor negative definite, by finding its pivots (and also its eigenvalues). Does

$$[x_1 \quad x_2] \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + 2x_1 x_2$$

have a minimum or maximum or saddle point at $x_1 = x_2 = 0$?

2.1.9 Given a network and its incidence matrices (ungrounded and grounded):



$$A_0 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(i) find $A^T C A$ from equations (5) and (6), with $-c_1$, $-c_2$, $-c_3$ appearing off the diagonal

(ii) find A^TCA from "column-row" multiplications, where the first column of A^T and the first row of A give

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} [c_1][-1 \quad 1 \quad 0] = \begin{bmatrix} c_1 & -c_1 & 0 \\ -c_1 & c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Add up the five products of this kind (one for each edge).

2.1.10 Eliminate y to find the equations for x in the systems

$$\begin{bmatrix} C^{-1} & A \\ A^T & D \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} C_1^{-1} & 0 & A_1 \\ 0 & C_2^{-1} & A_2 \\ A_1^T & A_2^T & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix}.$$

2.1.11 What is the equation for the vector x that minimizes $P(x) = \frac{1}{2}(b - Ax)^T C(b - Ax)$?

2.1.12 Draw a network with no loops (a *tree*). Check that with one node grounded the incidence matrix A is square, and find A^{-1}. All entries of the inverse are 1, -1, or 0.

2.1.13 Suppose A_0 is a 12 by 9 incidence matrix from a connected graph. Its exact form is unknown but it has 12 edges and 9 nodes, none grounded.

(a) How many columns of A_0 are independent?

(b) How many rows are independent, and what do the corresponding edges look like on the graph?

(c) What condition on f makes $A_0^T y = f$ solvable?

(d) How many independent solutions are there to $A_0^T y = 0$, and how can they be found from the graph? (See page 74.)

(e) For which vectors b does $A_0 x = b$ have a solution?